

Linear Systems I

Lecture 5

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Reading suggestion: Lecture 5 of Ref [2]. Ch 4.3, 4.4, 4.5 from Ref[1] ([1] treats the state transition matrix derivation in a different way than what we did in the class.

- Algebraically equivalent LTI systems
- Solution of LTV systems
 - Solution to Homogeneous Linear systems
 - Transition matrix and its properties

Note: This note only contains part of the material discussed in the class. For further details see your class notes.

Def(Zero-state equivalence): Two state-space systems are said to be zero-state equivalent if they realize the same transfer function, which means that they exhibit the same forced-response to every input. Zero-state equivalent systems does not necessarily are of the same dimension.

Def(Algebraically equivalent) Two continuous-time LTI systems

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \quad \text{and} \quad \begin{cases} \dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t), \\ y(t) = \bar{C}\bar{x}(t) + \bar{D}u(t), \end{cases}$$

are called algebraically equivalent if and only if there exists a nonsingular T s. t. ($\bar{A} = TAT^{-1}$, $\bar{B} = TB$, $\bar{C} = CT^{-1}$, $\bar{D} = D$). The corresponding map $\bar{x} = Tx$ is called a similarity transformation or an equivalence transformation.

Properties of two algebraically equivalent LTI systems

- P1.** With every input signal u , both systems associate the same set of outputs y . However, the output is generally not the same for the same initial conditions, except for the forced or zero-state response, which is always the same.
- P2.** the systems are zero-state equivalent, i.e., they have the same transfer function.

$$\begin{aligned}\bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D} &= C(sI - A)^{-1}B + D \\ \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D} &= CT^{-1}(sI - TAT^{-1})^{-1}TB + D = \\ &= CT^{-1}(sTT^{-1} - TAT^{-1})^{-1}TB + D = \\ &= CT^{-1}(T(sI - A)^{-1}T^{-1})TB + D = \\ &= C(sI - A)^{-1}B + D.\end{aligned}$$

Attention: In general the converse of P2. does not hold, i.e., zero-state equivalence does not imply algebraic equivalence. For two state equations to be equivalent, they must have the same dimension. This is, however, is not required for zero-state equivalent systems.

- P3.** they have the same eigenvalues.¹

$$\bar{\Delta}(\lambda) = \det(\lambda I - \bar{A}) = \det(\lambda I - A) = \Delta(\lambda)$$

The equivalent state equations have the same characteristic polynomial and consequently the same set of eigenvalues.

$$\begin{aligned}\bar{\Delta}(\lambda) = \det(\lambda I - \bar{A}) &= \det(\lambda TT^{-1} - TAT^{-1}) = \det(T) \det(\lambda I - A) \det(T^{-1}) = \\ &= \det(\lambda I - A) \det(T) \det(T^{-1}) = \det(\lambda I - A) = \Delta(\lambda).\end{aligned}$$

¹recall $\det(AB) = \det(A) \det(B) = \det(B) \det(A)$

We want to study the properties of solutions to SS LTV systems

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),$$

$$y(t) = C(t)x(t) + D(t)u(t),$$

$A(t) : [0, \infty) \rightarrow \mathbb{R}^{n \times n}$, $B(t) : [0, \infty) \rightarrow \mathbb{R}^{n \times p}$, $C(t) : [0, \infty) \rightarrow \mathbb{R}^{q \times n}$,
 $D(t) : [0, \infty) \rightarrow \mathbb{R}^{q \times p}$. Start by study of

homogeneous linear system: $\dot{x} = A(t)x(t)$, $x(t_0) = x_0 \in \mathbb{R}^n$, $t \geq t_0$. (1)

Theorem (Peano-Baker Series). The unique solution to (1) is given by

$$x(t) = \phi(t, t_0)x_0, \quad (2)$$

$$\phi(t, t_0) = I + \int_{t_0}^t A(\tau_1) d\tau_1 + \int_{t_0}^t A(\tau_1) \int_{t_0}^{\tau_1} A(\tau_2) d\tau_2 d\tau_1 + \int_{t_0}^t A(\tau_1) \int_{t_0}^{\tau_1} A(\tau_2) \int_{t_0}^{\tau_2} A(\tau_3) d\tau_3 d\tau_2 d\tau_1 + \dots$$

- $\phi(t, t_0)$: transition matrix (size $n \times n$)
- The series above is called Peano-Baker series

Properties of $\phi(t, t_0)$

P1. For every t_0 , $\phi(t, t_0)$ is the unique solution of

$$\frac{d}{dt}\phi(t, t_0) = A(t)\phi(t, t_0), \quad \phi(t_0, t_0) = I, \quad t \geq t_0.$$

P2 For every fixed t_0 , the i^{th} column of $\phi(t, t_0)$ is the unique solution to

$$\dot{x} = A(t)x(t), \quad x(t_0) = e_i, \quad t \geq t_0,$$

where e_i is the i^{th} column of identity matrix I_n , or equivalently a column vector of all zero entries except for the i^{th} which is equal to 1.

P3. For every t, s, τ we have

$$\phi(t, s)\phi(s, \tau) = \phi(t, \tau).$$

This property is called the semigroup property.

P4. For every t, τ , $\phi(t, t_0)$, is nonsingular and

$$\phi(t, \tau)^{-1} = \phi(\tau, t).$$

From P3 we have $\phi(t, \tau)\phi(\tau, t) = \phi(t, t)$ which gives $\phi(t, \tau)\phi(\tau, t) = I$. From P3 we can also write $\phi(\tau, t)\phi(t, \tau) = \phi(\tau, \tau)$ which gives $\phi(\tau, t)\phi(t, \tau) = I$. Therefore we have $\phi(t, \tau)\phi(\tau, t) = \phi(\tau, t)\phi(t, \tau) = I$. This completes the proof (recall the definition of an inverse of a matrix).

Note: Here, we used $\phi(t, t) = I$ for all t .

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$$A(t) : [0, \infty) \rightarrow \mathbb{R}^{n \times n}, \quad B(t) : [0, \infty) \rightarrow \mathbb{R}^{n \times p}, \quad C(t) : [0, \infty) \rightarrow \mathbb{R}^{q \times n}, \\ D(t) : [0, \infty) \rightarrow \mathbb{R}^{q \times p}.$$

Theorem (Variation of constants): The unique solution to LTV SS equation above is given by

$$x(t) = \phi(t, t_0)x_0 + \int_{t_0}^t \phi(t, \tau)B(\tau)u(\tau)d\tau$$

$$y(t) = C(t)\phi(t, t_0)x_0 + \int_{t_0}^t C(t)\phi(t, \tau)B(\tau)u(\tau)d\tau + D(t)u(t),$$

where $\phi(t, t_0)$ is the state transition matrix (as defined before).

$$y(t) = \underbrace{C(t)\phi(t, t_0)x_0}_{\text{homogeneous response}} + \underbrace{\int_{t_0}^t C(t)\phi(t, \tau)B(\tau)u(\tau)d\tau + D(t)u(t)}_{\text{forced response}}.$$