Linear Systems I
Lecture 4

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Reading material: Ch 4.4, pages 100-106, from Ref[1]. Lecture 4 of Ref[2].

Today's lecture

- Review of Impulse Response and Transfer Function
- From LTI State-Space form to Transfer Function
- From Transfer Function to LTI State-Space form

This note contains only part of the material discussed in the class.
For more details see your class notes.

Elementary Realization (from TF rep. to SS rep.)

Def. (Realization problem): how to compute SS representation from a given transfer function.

- Note every TF is not necessarily realizable. Recall that distributed systems have impulse response and as a result transfer function but no SS rep.

Def. (Realizable TF): A transfer function $\hat{G}(s)$ is said to be realizable if there exists a finite dimensional SS equation

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t) + Du(t),\end{aligned}$$

or simply $\{A, B, C, D\}$ such that

$$\hat{G}(s) = C(sI - A)^{-1}B + D.$$

We call $\{A, B, C, D\}$ a realization of $\hat{G}(s)$.

Note: if a transfer function is realizable it has infinitely many realization, not necessarily of the same dimension.

Realizable TFs

Theorem (realizable transfer function: A transfer function $\hat{G}(s)$ can be realized by an LTI SS equation iff ($\Leftrightarrow$) $\hat{G}(s)$ is a proper rational function.

Proof:

- ($\Rightarrow$) from a SS realization to a proper rational TF
- ($\Leftarrow$) from a proper rational TF to a SS realization

From rational proper TF to SS

Consider a linear causal system with p inputs and q outputs.

a proper TF $\hat{G}(s) \Rightarrow$ **SS representation**:
find $\{A, B, C, D\}$ such that $\hat{G}(s) = C(sI - A)^{-1}B + D$.

- Write $\hat{G}(s) = \hat{G}_{sp}(s) + D$, where $D = \lim_{s\to\infty} \hat{G}(s)$
- Find the monic least comment denominator of all the entries of $\hat{G}_{sp}(s)$ matrix,

  $$d(s) = 1s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n.$$

  The monic least common denominator of a family of polynomials is the monic polynomial of the smallest order that can be divided by all those polynomials.
- $\hat{G}_{sp}(s) = \frac{1}{d(s)}[N_1 s^{n-1} + N_2 s^{n-2} + \cdots + N_{n-1}s + N_n]$,

From rational proper TF to SS

- $\hat{G}_{sp}(s) = \frac{1}{d(s)}[N_1 s^{n-1} + N_2 s^{n-2} + \cdots + N_{n-1}s + N_n]$,

We claim (controllable canonical form)

$$A = \begin{bmatrix} -\alpha_1 I_{p\times p} & -\alpha_2 I_{p\times p} & \cdots & -\alpha_{n-1} I_{p\times p} & -\alpha_n I_{p\times p} \\ I_{p\times p} & 0_{p\times p} & \cdots & 0_{p\times p} & 0_{p\times p} \\ 0_{p\times p} & I_{p\times p} & \cdots & 0_{p\times p} & 0_{p\times p} \\ \vdots & & \ddots & \vdots & \vdots \\ 0_{p\times p} & 0_{p\times p} & \cdots & I_{p\times p} & 0_{p\times p} \end{bmatrix}_{np\times np}, B = \begin{bmatrix} I_{p\times p} \\ 0_{p\times p} \\ \vdots \\ 0_{p\times p} \\ 0_{p\times p} \end{bmatrix}_{np\times p},$$

$$C = \begin{bmatrix} N_1 & N_2 & \cdots & N_{n-1} & N_n \end{bmatrix}_{q\times np}, \quad D = \lim_{s\to\infty} \hat{G}(s)$$

We need to show
$\hat{G}_{sp}(s) = \frac{1}{d(s)}[N_1 s^{n-1} + N_2 s^{n-2} + \cdots + N_{n-1}s + N_n] = C(sI - A)^{-1}B$

show $\hat{G}_{sp}(s) = \frac{1}{d(s)}[N_1 s^{n-1} + N_2 s^{n-2} + \cdots + N_{n-1}s + N_n] = C(sI - A)^{-1}B$

- ⋆ Let $Z = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ \vdots \\ Z_n \end{bmatrix} = (sI - A)^{-1}B$
- ⋆ Then, $\hat{G}_{sp}(s) = C(sI - A)^{-1}B = CZ = N_1 Z_1 + N_2 Z_2 + \cdots + N_n Z_n$
- ⋆ Compare with $\hat{G}_{sp}(s) = \frac{1}{d(s)}[N_1 s^{n-1} + N_2 s^{n-2} + \cdots + N_{n-1}s + N_n]$
- ⋆ Therefore, we need to show that

$$Z_1 = \frac{s^{n-1}}{d(s)} I_{p\times p},\ Z_2 = \frac{s^{n-2}}{d(s)} I_{p\times p}, \cdots, Z_{n-1} = \frac{s}{d(s)} I_{p\times p},\ Z_n = \frac{1}{d(s)} I_{p\times p}.$$

show $\hat{G}_{sp}(s) = \frac{1}{d(s)}[N_1 s^{n-1} + N_2 s^{n-2} + \cdots + N_{n-1}s + N_n] = C(sI - A)^{-1}B$

$$Z_1 = \frac{s^{n-1}}{d(s)} I_{p\times p},\ Z_2 = \frac{s^{n-2}}{d(s)} I_{p\times p}, \cdots, Z_{n-1} = \frac{s}{d(s)} I_{p\times p},\ Z_n = \frac{1}{d(s)} I_{p\times p}.$$

which can be deduced from the following

$$Z = \begin{bmatrix} Z_1 \\ Z_2 \\ \cdots \\ Z_n \end{bmatrix} = (sI - A)^{-1}B \Rightarrow (sI - A)Z = B \Rightarrow sZ = AZ + B$$

$$\begin{bmatrix} sZ_1 \\ sZ_2 \\ sZ_3 \\ \vdots \\ sZ_n \end{bmatrix} = \begin{bmatrix} -\alpha_1 I_{p\times p} & -\alpha_2 I_{p\times p} & \cdots & -\alpha_{n-1} I_{p\times p} & -\alpha_n I_{p\times p} \\ I_{p\times p} & 0_{p\times p} & \cdots & 0_{p\times p} & 0_{p\times p} \\ 0_{p\times p} & I_{p\times p} & \cdots & 0_{p\times p} & 0_{p\times p} \\ \vdots & & \ddots & \vdots & \vdots \\ 0_{p\times p} & 0_{p\times p} & \cdots & I_{p\times p} & 0_{p\times p} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ \vdots \\ Z_n \end{bmatrix} + \begin{bmatrix} I_{p\times p} \\ 0_{p\times p} \\ 0_{p\times p} \\ \vdots \\ 0_{p\times p} \end{bmatrix} \Rightarrow$$

show $\hat{G}_{sp}(s) = \frac{1}{d(s)}[N_1 s^{n-1} + N_2 s^{n-2} + \cdots + N_{n-1}s + N_n] = C(sI - A)^{-1}B$

$$\begin{bmatrix} sZ_1 \\ sZ_2 \\ sZ_3 \\ \vdots \\ sZ_n \end{bmatrix} = \begin{bmatrix} -\alpha_1 I_{p\times p} & -\alpha_2 I_{p\times p} & \cdots & -\alpha_{n-1} I_{p\times p} & -\alpha_n I_{p\times p} \\ I_{p\times p} & 0_{p\times p} & \cdots & 0_{p\times p} & 0_{p\times p} \\ 0_{p\times p} & I_{p\times p} & \cdots & 0_{p\times p} & 0_{p\times p} \\ \vdots & & \ddots & \vdots & \vdots \\ 0_{p\times p} & 0_{p\times p} & \cdots & I_{p\times p} & 0_{p\times p} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ \vdots \\ Z_n \end{bmatrix} + \begin{bmatrix} I_{p\times p} \\ 0_{p\times p} \\ 0_{p\times p} \\ \vdots \\ 0_{p\times p} \end{bmatrix} \Rightarrow$$

$$\begin{cases} s\,Z_1 = -\alpha_1 Z_1 - \alpha_2 Z_2 - \cdots - \alpha_{n-1} Z_{n-1} - \alpha_n Z_n + I_{p\times p} \\ s\,Z_2 = Z_1 \Rightarrow Z_2 = \frac{1}{s} Z_1 \\ s\,Z_3 = Z_2 \Rightarrow Z_3 = \frac{1}{s^2} Z_1 \\ \vdots \\ s\,Z_n = Z_{n-1} \Rightarrow Z_n = \frac{1}{s^{n-1}} Z_1 \end{cases} \Rightarrow \begin{cases} Z_1 = \frac{s^{n-1}}{d(s)} I_{p\times p} \\ \quad \Downarrow \\ Z_2 = \frac{s^{n-2}}{d(s)} I_{p\times p} \\ \vdots \\ Z_n = \frac{1}{d(s)} I_{p\times p} \end{cases}$$

$$d(s) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n.$$

show $\hat{G}_{sp}(s) = \frac{1}{d(s)}[N_1 s^{n-1} + N_2 s^{n-2} + \cdots + N_{n-1}s + N_n] = C(sI - A)^{-1}B$

Let $Z = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ \vdots \\ Z_n \end{bmatrix} = (sI - A)^{-1}B$ Then, $\hat{G}_{sp}(s) = C(sI - A)^{-1}B = CZ = N_1 Z_1 + N_2 Z_2 + \cdots + N_n Z_n$

Therefore, we need to show that

$$Z_1 = \frac{s^{n-1}}{d(s)} I_{p\times p},\ Z_2 = \frac{s^{n-2}}{d(s)} I_{p\times p}, \cdots, Z_{n-1} = \frac{s}{d(s)} I_{p\times p},\ Z_n = \frac{1}{d(s)} I_{p\times p}.$$

which can be deduced from the following (recall that $d(s) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n$):

$$Z = \begin{bmatrix} Z_1 \\ Z_2 \\ \cdots \\ Z_n \end{bmatrix} = (sI - A)^{-1}B \Rightarrow (sI - A)Z = B \Rightarrow sZ = AZ + B$$

$$\begin{bmatrix} sZ_1 \\ sZ_2 \\ sZ_3 \\ \vdots \\ sZ_n \end{bmatrix} = \begin{bmatrix} -\alpha_1 I_{p\times p} & -\alpha_2 I_{p\times p} & \cdots & -\alpha_{n-1} I_{p\times p} & -\alpha_n I_{p\times p} \\ I_{p\times p} & 0_{p\times p} & \cdots & 0_{p\times p} & 0_{p\times p} \\ 0_{p\times p} & I_{p\times p} & \cdots & 0_{p\times p} & 0_{p\times p} \\ \vdots & & \ddots & \vdots & \vdots \\ 0_{p\times p} & 0_{p\times p} & \cdots & I_{p\times p} & 0_{p\times p} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ \vdots \\ Z_n \end{bmatrix} + \begin{bmatrix} I_{p\times p} \\ 0_{p\times p} \\ 0_{p\times p} \\ \vdots \\ 0_{p\times p} \end{bmatrix} \Rightarrow$$

$$\begin{cases} sZ_1 = -\alpha_1 Z_1 - \alpha_2 Z_2 - \cdots - \alpha_{n-1} Z_{n-1} - \alpha_n Z_n + I_{p\times p} \\ sZ_2 = Z_1 \Rightarrow Z_2 = \frac{1}{s} Z_1 \\ sZ_3 = Z_2 \Rightarrow Z_3 = \frac{1}{s^2} Z_1 \\ \vdots \\ sZ_n = Z_{n-1} \Rightarrow Z_n = \frac{1}{s^{n-1}} Z_1 \end{cases} \Rightarrow \begin{cases} Z_1 = \frac{s^{n-1}}{d(s)} I_{p\times p} \\ \quad \Downarrow \\ Z_2 = \frac{s^{n-2}}{d(s)} I_{p\times p} \\ \vdots \\ Z_n = \frac{1}{d(s)} I_{p\times p} \end{cases}$$

Example: from transfer function to state space representation

$$\hat{G}(s) = \begin{bmatrix} \frac{s+1}{s+3} \\ \frac{s-1}{s+1} \\ \frac{s+2}{(s+1)(s+3)} \end{bmatrix}, \quad \text{this system has } p = 1 \text{ input and } q = 3 \text{ output.}$$

- Write $\hat{G}(s) = \hat{G}_{sp}(s) + D$, where $D = \lim_{s\to\infty} \hat{G}(s)$.

$$D = \lim_{s\to\infty} \hat{G}(s) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \hat{G}_{sp}(s) = \hat{G}(s) - D = \begin{bmatrix} \frac{s+1}{s+3} \\ \frac{s-1}{s+1} \\ \frac{s+2}{(s+1)(s+3)}, \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{2}{s+3} \\ -\frac{2}{s+1} \\ \frac{s+2}{(s+1)(s+3)} \end{bmatrix}.$$

- Find the monic least comment denominator of all the entries of $\hat{G}_{sp}(s)$ matrix,

$$d(s) = 1s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n.$$

$$d(s) = (s+1)(s+2)(s+3) = s^3 + \underbrace{6}_{\alpha_1} s^2 + \underbrace{11}_{\alpha_2} s + \underbrace{6}_{\alpha_3}.$$

- Write $\hat{G}_{sp}(s) = \frac{1}{d(s)}[N_1 s^{n-1} + N_2 s^{n-2} + \cdots + N_{n-1}s + N_n]$

$$\hat{G}_{sp}(s) = \frac{1}{s^3+6s^2+11s+6} \begin{bmatrix} -2(s+1)(s+2) \\ -2(s+2)(s+3) \\ (s+2)(s+2) \end{bmatrix} = \frac{1}{s^3+6s^2+11s+6} \left(\underbrace{\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}}_{N_1} s^2 + \underbrace{\begin{bmatrix} -6 \\ -10 \\ 4 \end{bmatrix}}_{N_2} s + \underbrace{\begin{bmatrix} -4 \\ -12 \\ 4 \end{bmatrix}}_{N_3} \right)$$

$$A = \begin{bmatrix} -6 & -11 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & -6 & -4 \\ -2 & -10 & -12 \\ 1 & 4 & 4 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Zero-state equivalence and algebraically equivalence

Def(Zero-state equivalence): Two state-space systems are said to be zero-state equivalent if they realize the same transfer function, which means that they exhibit the same forced-response to every input. Zero-state equivalent systems does <u>not</u> necessarily are of the same dimension. The following SS forms are zero-state equivalent.

$$A = \left[\begin{array}{cc|cc|cc} -4.5 & 0 & -6 & 0 & -2 & 0 \\ 0 & -4.5 & 0 & -6 & 0 & -2 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right], \quad B = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right]$$

$$C = \left[\begin{array}{cc|cc|cc} -6 & 3 & -24 & 7.5 & -24 & 3 \\ 0 & 1 & 0.5 & 1.5 & 1 & 0.5 \end{array}\right], \quad D = \left[\begin{array}{cc} 2 & 0 \\ 0 & 0 \end{array}\right]$$

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$$\bar{A} = \left[\begin{array}{cccc} -2.5 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -4 & -4 \\ 0 & 0 & 1 & 0 \end{array}\right], \quad \bar{B} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{array}\right]$$

$$\bar{C} = \left[\begin{array}{cccc} -6 & -12 & 3 & 6 \\ 0 & 0.5 & 1 & 1 \end{array}\right], \quad \bar{D} = \left[\begin{array}{cc} 2 & 0 \\ 0 & 0 \end{array}\right]$$

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$$C(sI - A)^{-1}B + D = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D}$$

Algebraically equivalent LTI systems

Consider

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$

Given T nonsingular, apply change of variable $\bar{x} = Tx$ to write the system in the new state $\bar{x}$

$$\begin{cases} \dot{\bar{x}} = T\dot{x} = T(Ax(t) + Bu(t)) = \underbrace{TAT^{-1}}_{\bar{A}}\,\bar{x} + \underbrace{TB}_{\bar{B}}\,u(t) \\ y(t) = C(t)x(t) + D(t)u(t) = \underbrace{CT^{-1}}_{\bar{C}}\,\bar{x} + \underbrace{D}_{\bar{D}}\,u(t) \end{cases} \Rightarrow \begin{cases} \dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t), \\ y(t) = \bar{C}\bar{x}(t) + \bar{D}u(t), \end{cases}$$

Def(Algebraically equivalent) Two continuous-time LTI systems

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \quad \text{and} \quad \begin{cases} \dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t), \\ y(t) = \bar{C}\bar{x}(t) + \bar{D}u(t), \end{cases}$$

are called algebraically equivalent if and only if there exists a nonsingular T s. t. ($\bar{A} = TAT^{-1}$, $\bar{B} = TB$, $\bar{C} = CT^{-1}$, $\bar{D} = D$). The corresponding map $\bar{x} = Tx$ is called a similarity transformation or an equivalence transformation.