Optimal Control
Lecture 14

Solmaz S. Kia
Mechanical and Aerospace Engineering Dept.
University of California Irvine
solmaz@uci.edu

Note: These slides only cover small part of the lecture. For details and other discussions consult your class notes.
Reading suggestion: Section 5.2 of Ref [1]
For infinite horizon LQR you can study: Section 6.2 of Ref[3] (see the syllabus/class website for the list of the references)".

Outline

- Optimal Control
  - Properties of optimal contorl
  - LQR problem for continuous-time systems
  - Dynamic signal tracking in finite-time

Optimal control (Review)

$$u^\star(t)\Big|_{t\in[t_0,t_f]} = \underset{u(t)\in\mathcal{U}}{\text{argmin}}\left(J = h(x(t_f),t_f) + \int_{t_0}^{t_f} g(x(t),u(t),t))dt,\quad s.t.\right.$$

$$\dot{x}(t) = a(x(t),u(t),t),$$

$x(t_0)$, t_0 is given,

$m(x(t_f),t_f) = 0 \leftarrow$ when final state is constrained,

$x(t):\mathbb{R}\to\mathbb{R}^n,\quad u(t):\mathbb{R}\to\mathbb{R}^m,\quad f:\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}\to\mathbb{R}^n.$

- Hamiltonian $H(x,u,p,t) = g(x(t),u(t),t) + p(t)^\top a(x(t),u(t),t),$

first order conditions for extremal solution

$$\dot{p} = -H_x, \qquad (n \text{ dimensional})$$

$$0 = H_u, \qquad (m \text{ dimensional})$$

$$\dot{x} = H_p: \quad \dot{x} = a(x,u,t), \qquad (n \text{ dimensional})$$

— — — — — — — — — — — — — boundary conditions — — — — — — — — — — — — —

- $x(t_0) = x_0$
- if t_f free: $\frac{\partial h}{\partial t}\Big|_{t_f} + H(t_f) = 0$
- if $x_i(t_f)$ is fixed: $x_i(t_f) = x_{i_f}$
- if $x_i(t_f)$ is free: $p_i(t_f) = \frac{\partial h}{\partial x_i}(t_f)$

$m(x(t_f),t_f) = 0$

Let $w(x(t_f),\nu,t_f) = h(x(t_f),t_f) + \nu^\top m(x(t_f),t_f)$

- $x(t_0) = x_0$
- since $x(t_f)$ is not directly given we need $p(t_f) = \frac{\partial w}{\partial x}(t_f)$
- if t_f free: $\frac{\partial w}{\partial t}\Big|_{t_f} + H(t_f) = 0$ (disappears if t_f known)

Properties of Hamiltonian

$$u^\star(t)\Big|_{t\in[t_0,t_f]} = \underset{u(t)\in\mathcal{U}}{\text{argmin}}(J = h(x(t_f),t_f) + \int_{t_0}^{t_f} g(x(t),u(t),t))dt, \;\; s.t.$$

$\dot{x}(t) = a(x(t),u(t),t),$

$x(t_0),\;\; t_0$ is given,

$x(t_f),\;\; t_f$various final state constrained,

$x(t):\mathbb{R}\to\mathbb{R}^n, \quad u(t):\mathbb{R}\to\mathbb{R}^m, \quad f:\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}\to\mathbb{R}^n,\;\; C:\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}\to\mathbb{R}^q.$

$$H = g(x(t),u(t),t)) + p(t)^\top a(x(t),u(t),t)$$

- If $g(x,u)$ and $a(x,u)$ do not explicitly depend on time t, then the Hamiltonian H is at least piece-wise contract.

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + (\frac{\partial H}{\partial x})\frac{dx}{dt} + (\frac{\partial H}{\partial p})\frac{dp}{dt} + (\frac{\partial H}{\partial u})\frac{du}{dt}$$

$$\frac{dH}{dt} = H_x\dot{x} + H_p\dot{p} + H_u\dot{u}$$

From F.O.N condition : $\dot{x} = H_p$ and $\dot{p} = -H_x$, then

$$\frac{dH}{dt} = H_u\dot{u}$$

$$\frac{dH}{dt} = H_u \dot{u}$$

- The third necessary condition is $H_u = 0$ so

$$\frac{dH}{dt} = (0)\dot{u} = 0$$

  which suggest H is constant

  - It might be possible for the value of this constant to change at a discontinuity of u, since then $\dot{u}$ would be infinite, and $0.\infty$ is not defined.
  - This H is at least piece-wise constant.

- For free final time problems, the transversality condition gives

$$h_t + H(t_f) = 0$$

- If h is not function of time then $h_t = 0$ and as a result $H(t_f) = 0$.
- with no jumps in u, H is constant: $H(t) = 0$ for all $t \in [t_0.t_f]$.

Finite time LQR problem

$$u^\star(t)\Big|_{t\in[t_0,t_f]} = \underset{u(t)\in\mathcal{U}}{\text{argmin}}(J = \frac{1}{2}x(t_f)^\top H^f x(t_f) + \frac{1}{2}\int_0^{t_f}(x^\top Q(t)x + u^\top R(t)u)dt, \ \ \text{s.t.}$$

$$\dot{x}(t) = A(t)x + B(t)u,$$

$x(t_0)$, t_0 is given,

$x(t_f)$ is free

$$x(t):\mathbb{R}\to\mathbb{R}^n, \quad u(t):\mathbb{R}\to\mathbb{R}^m, \quad Q(t)\geqslant 0,\ H^f\geqslant 0,\ R(t) > 0.$$

- Hamiltonian

$$H(x,u,p,t) = \frac{1}{2}\Big(x(t)^\top Q(t)x(t) + u^\top(t)R(t)u(t)\Big) + p(t)^\top(A(t)x(t) + B(t)u(t)),$$

FONC:

- $\dot{x} = H_p \quad \Rightarrow \quad \dot{x} = A(t)x + B(t)u$
- $\dot{p} = -H_x \Rightarrow \quad \dot{p} = -Q(t)x - A(t)^\top p,$
- $0 = H_u \quad \Rightarrow \quad R(t)u(t) + B(t)^\top p(t) = 0 \quad \Rightarrow u^\star(t) = -R(t)^{-1}B(t)^\top p(t)^\star$
- Boundary condition

$$p(t_f) = \frac{\partial \frac{1}{2}x^\top H^f x}{\partial x}\Big|_{t_f} = H^f x(t_f)$$

Finite time LQR

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A(t) & -B(t)R(t)^{-1}B(t)^\top \\ -Q(t) & -A(t)^\top \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}$$

with BC

$$\begin{cases} x(0) = x_0, \\ p(t_f) = H^f x(t_f) \end{cases}$$

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$$\begin{bmatrix} x(t_f) \\ p(t_f) \end{bmatrix} = \underbrace{\begin{bmatrix} \phi_{11}(t_f,t) & \phi_{12}(t_f,t) \\ \phi_{21}(t_f,t) & \phi_{22}(t_f,t) \end{bmatrix}}_{\phi(t_f,t)} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix}$$

$$\Downarrow$$

$$p^\star(t) = \underbrace{\big(\phi_{22}(t_f,t) - H^f\phi_{12}(t_f,t)\big)^{-1}\big(H^f\phi_{11}(t_f,t) - \phi_{21}(t_f,t)\big)}_{K(t)} x^\star(t)$$

$$p^\star(t) = K(t)x^\star(t)$$

Finite time LQR

$$p^\star(t) = K(t)x^\star(t)$$

$$u^\star(t) = -R(t)^{-1}B(t)^\top p^\star(t)$$

$$\boxed{u^\star = -R(t)^{-1}B(t)^\top K(t)\,x^\star(t)}$$

$$K(t) = \big(\phi_{22}(t_f,t) - H^f\phi_{12}(t_f,t)\big)^{-1}\big(H^f\phi_{11}(t_f,t) - \phi_{21}(t_f,t)\big)$$

or

Riccati type differential equation: $\dot{p}(t) = \dot{K}(t)x(t) + K(t)\dot{x}(t) \Rightarrow$

$$\begin{cases} \dot{K}(t) = -K(t)A(t) - A^\top(t)K(t) - Q(t) + K(t)B(t)R^{-1}(t)B^\top(t)K(t), \\ K(t_f) = H^f. \end{cases}$$

- Optimal tracking control is a state feedback control
- The state feedback gain is time-varying
- Optimal cost to go $J_{i\to N}(x(t_i)) = \frac{1}{2}x(t_i)^\top K(t_i)x(t_i)$.

LQR: the steady state Riccati equation (SSRE)

For LTI systems, Kalman has shown that the Riccati equation has a steady state solution if (see Kirk)

- A, B, R, Q are constant
- $H^f = 0$,
- (A, B) is controllable,

SS Riccati equation is $A^\top K + KA + Q - KBR^{-1}B^\top K = 0$

- You can find the SS feedback using $F = \mathrm{lqr}(A, B, Q, R)$ in Matlab. For further information see https://www.mathworks.com/help/control/ref/lqr.html

Finite time linear tracking problem

$$u^\star(t)\Big|_{t\in[t_0,t_f]} = \underset{u(t)\in\mathcal{U}}{\operatorname{argmin}}\Big(J = \frac{1}{2}(x(t_f)-r(t_f))^\top H^f(x(t_f)-r(t_f)) + \frac{1}{2}\int_0^{t_f}((x(t)-r(t))^\top Q(t)(x(t)-r(t)) + u^\top R(t)u)dt, \quad \text{s.t.} \quad s$$

$\dot{x}(t) = A(t)x + B(t)u,$

$x(t_0)$, t_0 is given,

$x(t_f)$ is free

$x(t):\mathbb{R}\to\mathbb{R}^n, \quad u(t):\mathbb{R}\to\mathbb{R}^m, \quad Q(t)\geqslant 0,\ H^f\geqslant 0,\ R(t)>0.$

- Hamiltonian

$$H(x,u,p,t) = \frac{1}{2}\Big((x(t)-r(t))^\top Q(t)(x(t)-r(t)) + u^\top(t)R(t)u(t)\Big) + p(t)^\top(A(t)x(t)+B(t)u(t)),$$

FONC:

- $\dot{x} = H_p \quad \Rightarrow \quad \dot{x} = A(t)x + B(t)u$
- $\dot{p} = -H_x \Rightarrow \quad \dot{p} = -Q(t)x - A(t)^\top p,$
- $0 = H_u \quad \Rightarrow \quad R(t)u(t) + B(t)^\top p(t) = 0 \quad \Rightarrow u^\star(t) = -R(t)^{-1}B(t)^\top p(t)^\star$
- Boundary condition: $p(t_f) = \frac{\partial \frac{1}{2}(x-r(t_f))^\top H^f(x-r(t_f))}{\partial x}\Big|_{t_f} = H^f(x(t_f)-r(t_f))$

Finite time linear tracking problem

$$\begin{bmatrix}\dot{x}\\ \dot{p}\end{bmatrix} = \begin{bmatrix}A(t) & -B(t)R(t)^{-1}B(t)^\top \\ -Q(t) & -A(t)^\top\end{bmatrix}\begin{bmatrix}x\\ p\end{bmatrix} + \begin{bmatrix}0\\ Q(t)r(t)\end{bmatrix}$$

with BC

$$\begin{cases} x(0) = x_0, \\ p(t_f) = H^f x(t_f) - H^f r(t_f) \end{cases}$$

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$$\begin{bmatrix}x(t_f)\\ p(t_f)\end{bmatrix} = \underbrace{\begin{bmatrix}\phi_{11}(t_f,t) & \phi_{12}(t_f,t)\\ \phi_{21}(t_f,t) & \phi_{22}(t_f,t)\end{bmatrix}}_{\phi(t_f,t)}\begin{bmatrix}x(t)\\ p(t)\end{bmatrix} + \underbrace{\int_{t_0}^{t_f} \phi(t_f,\tau)\begin{bmatrix}0\\ Q(\tau)r(\tau)\end{bmatrix} d\tau}_{\begin{bmatrix}f_1(t)\\ f_2(t)\end{bmatrix}}$$

$$+$$

$$p(t_f) = H^f x(t_f) - H^f r(t_f)$$

$$\Downarrow$$

$$p^\star(t) = K(t)x^\star(t) + s(t)$$

Finite time linear tracking problem

$$p^\star(t) = K(t)x^\star(t) + s(t)$$

$$u^\star(t) = -R(t)^{-1}B(t)^\top p(t)^\star$$

$$\boxed{u^\star = -R(t)^{-1}B(t)^\top K(t)\, x^\star(t) - R(t)^{-1}B(t)^\top s(t) = F(t)x^\star(t) + v(t)}$$

where $K(t)$ and $s(t)$ are computed from (use $\dot{p}(t) = \dot{K}(t)x(t) + K(t)\dot{x}(t) + \dot{s}(t)$)

$$\begin{cases} \dot{K} = -K(t)A(t) - A^\top(t)K(t) - Q(t) + K(t)B(t)R^{-1}(t)B^\top(t)K(t), & K(t_f) = H^f, \\ \dot{s} = -(A(t)^\top - K(t)B(t)R(t)^{-1}B(t)^\top)s(t) + Q(t)r(t), & s(t_f) = -H^f r(t_f). \end{cases}$$