3.26pt

Optimal Control
Lecture 13

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Optimal control

We are going to focus on solving

$$u^\star(t)\Big|_{t\in[t_0,t_f]} = \underset{u(t)\in\mathcal{U}}{\text{argmin}}\Big(J = h(x(t_f),t_f) + \int_{t_0}^{t_f} g(x(t),u(t),t)dt, \quad s.t.$$

$$\dot{x}(t) = a(x(t),u(t),t),$$

$$x(t_0), \ t_0 \text{ is given},$$

$$m(x(t_f),t_f) = 0 \leftarrow \text{ when final state is constrained},$$

$$x(t):\mathbb{R}\rightarrow\mathbb{R}^n, \quad u(t):\mathbb{R}\rightarrow\mathbb{R}^m, \quad f:\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}\rightarrow\mathbb{R}^n.$$

- Use Lagrange multiplier to write

$$J_a = h(x(t_f),t_f) + \int_{t_0}^{t_f}\Big(g(x(t),u(t),t) + p(t)^\top(a(x(t),u(t),t) - \dot{x}(t))\Big)dt$$

- Define the Hamiltonian to help with sorting out the equations

$$\boxed{H(x,u,p,t) = g(x(t),u(t),t) + p(t)^\top a(x(t),u(t),t),}$$

Optimal control

- $$\delta J_a = (h_x - p(t_f))^\top \delta x_f + \Big[h_{t_f} + g + p^\top (a - \dot{x}) + p^\top \dot{x}\Big]_{t_f} \delta t_f + \\ + \int_{t_0}^{t_f} \Big[(H_x + \dot{p})^\top \delta x(t) + H_u^\top \delta u(t) + (a - \dot{x})^\top \delta p(t)\Big] dt$$

first order conditions for extremal solution

$$\dot{p} = -H_x, \quad (n \text{ dimensional})$$
$$0 = H_u, \quad (m \text{ dimensional})$$
$$0 = H_p \rightarrow \quad \dot{x} = a(x, u, t), \quad (n \text{ dimensional})$$
$$\text{boundary condition } (h_x - p(t_f))^\top \delta x_f + \Big[h_{t_f} + g + p^\top a\Big]_{t_f} \delta t_f = 0$$

Boundary conditions $x(t_0) = x_0$, and

- if t_f free
$$h_{t_f} + g + p^\top a = h_{t_f} + H(t_f) = 0$$
- if $x_i(t_f)$ is fixed: $x_i(t_f) = x_{i_f}$
- if $x_i(t_f)$ is free, then $p_i(t_f) = \frac{\partial h}{\partial x_i}(t_f)$
- if t_f is free and $m(x(t_f), t_f) = 0$, (see next page)

Optimal control: when final time and state are related through $m(x(t_f), t_f) = 0$

- follow the same method as discussed for simplest problem in calculous of variation, write

$$w(x(t_f), t_f) = h(x(t_f), t_f) + \nu^\top m(x(t_f), t_f)$$

- work through the math to arrive at the following F.O.N conditions

first order conditions for extremal solution

$$\dot{p} = -H_x, \qquad (n \text{ dimensional})$$

$$0 = H_u, \qquad (m \text{ dimensional})$$

$$0 = H_p \rightarrow \quad \dot{x} = a(x, u, t), \qquad (n \text{ dimensional})$$

Boundary conditions $x(t_0) = x_0$, and $m(x(t_f), t_f) = 0$. Also

- if t_f free

$$w_f(t_f) + H_{t_f} = 0$$

-

$$p(t_f) = \frac{\partial w(t_f)}{\partial x}$$

Optimal control (Summary)

$$u^\star(t)\Big|_{t\in[t_0,t_f]} = \underset{u(t)\in\mathcal{U}}{\text{argmin}}\left(J = h(x(t_f),t_f) + \int_{t_0}^{t_f} g(x(t),u(t),t))dt, \quad s.t.\right.$$

$$\dot{x}(t) = a(x(t),u(t),t),$$

$$x(t_0),\ t_0 \text{ is given},$$

$$m(x(t_f),t_f) = 0 \leftarrow \text{ when final state is constrained},$$

$$x(t):\mathbb{R}\to\mathbb{R}^n, \quad u(t):\mathbb{R}\to\mathbb{R}^m, \quad f:\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}\to\mathbb{R}^n.$$

- Hamiltonian $H(x,u,p,t) = g(x(t),u(t),t) + p(t)^\top a(x(t),u(t),t)$,

first order conditions for extremal solution

$$\dot{p} = -H_x, \qquad (n \text{ dimensional})$$

$$0 = H_u, \qquad (m \text{ dimensional})$$

$$\dot{x} = H_p : \quad \dot{x} = a(x,u,t), \qquad (n \text{ dimensional})$$

— — — — — — — — — — — — — boundary conditions — — — — — — — — — — — — —

- $x(t_0) = x_0$
- if t_f free: $\frac{\partial h}{\partial t}\Big|_{t_f} + H(t_f) = 0$
- if $x_i(t_f)$ is fixed: $x_i(t_f) = x_{i_f}$
- if $x_i(t_f)$ is free: $p_i(t_f) = \frac{\partial h}{\partial x_i}(t_f)$

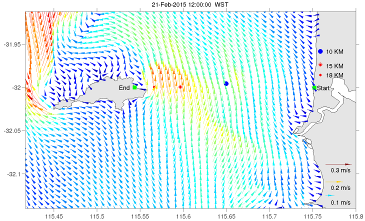
$m(x(t_f),t_f) = 0$

Let $w(x(t_f),\nu,t_f) = h(x(t_f),t_f) + \nu^\top m(x(t_f),t_f)$

- $x(t_0) = x_0$
- since $x(t_f)$ is not directly given we need $p(t_f) = \frac{\partial w}{\partial x}(t_f)$
- if t_f free: $\frac{\partial w}{\partial t}\Big|_{t_f} + H(t_f) = 0$ (disappears if t_f known)

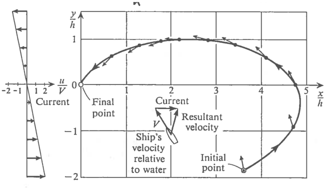
Constrained functional optimization: example

- Minimum-time path through a region of position dependent vector velocity (Zernelo's problem)



Example of an ocean current vector field

  - The forward velocity of the ship V is constant but its steering angle θ can be controlled.
  - in the depicted example, it is assume that the current's velocity vector is only in x direction



see [Bryson and Ho]

How to solve optimal control problems using Matlab

- You can use Matlab command 'bvp4c' to solve boundary value problems of the form

$$\dot{y} = f(y, t, p), \quad t \in [a, b],$$
$$g(y(a), y(b)) = 0, \quad \leftarrow \text{boundary conditions at } a, \text{ and } b.$$

  This is good for fixed and given final time problems.

- For further information visit:
  http://www.mathworks.com/help/matlab/ref/bvp4c.html

- For problems with free final time (t_f is a search variable too) we can use a trick to convert the problems to the standard from for 'bvp4c'

How to solve optimal control problems using Matlab (free final time)

- if $t_0 \neq 0$ transfer the initial time to zero, i.e., in our developments below we assume $t_0 = 0$
- use the scaling $\tau = \frac{t}{t_f}$ to normalize the time horizon from $[0, t_f]$ to $\tau \in [0, 1]$.
- you need to change all derivatives and express them in term of $d\tau$ instead of dt
- use $d\tau = \frac{1}{t_f} dt$ to write

$$\frac{d}{d\tau} = t_f \frac{d}{dt}$$

  - then introduce a new intermediate variable $\dot{s} = 0$ (to represent the constant t_f that we are searching)
  - in all boundary conditions you replace t_f with s
  - this way bvp4c returns the right value of t_f in s variable ($s = t_f$)
- The set of necessary conditions, for $t \in [0, t_f]$, are

$$\dot{x} = a(x, u, t),$$
$$\dot{p} = -H_x,$$
$$H_u = 0$$

You need to scale the time in these equations as follows

$$x' = t_f\, a(x, u, \tau), \qquad (x' = \frac{dx}{d\tau})$$
$$p' = -t_f\, H_x, \qquad (p' = \frac{dp}{d\tau})$$

see http://www4.ncsu.edu/~xwang10/document/Solving%20optimal%20control%20problems%20with%20MATLAB.pdf for further discussions.