Optimal Control
Lecture 1,2

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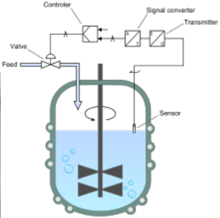

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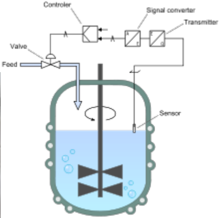

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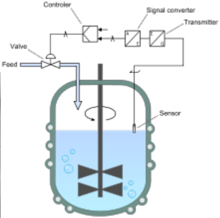

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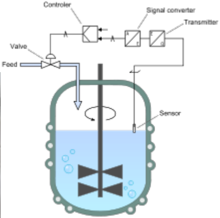

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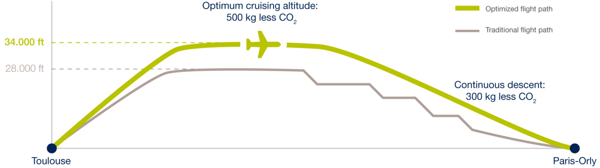

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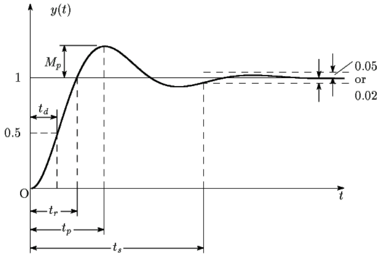
- Performance measures considering step or ramp response:

mostly for SISO systems

  - rise-time (t_r)
  - settling time (t_s)
  - peak overshoot (M_P)
  - gain and phase margin and bandwidth
  - steady state error



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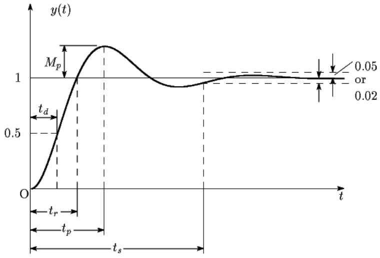

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  - more complex performance measures, perhaps more closely related to the physical aspects of the system
    - minimum fuel
    - minimum control effort
    - minimum time

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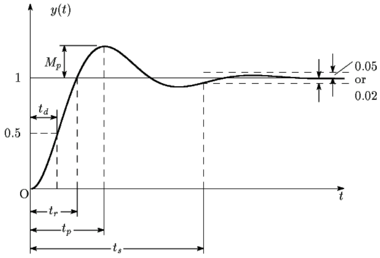

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    - minimum fuel
    - minimum control effort
    - minimum time
  - satisfy some constraints on control and states of the system while optimizing performance measure

The objective of optimal control is to determine the control signals that will cause a process to satisfy the physical constraints and at the same time minimize (or maximize) some performance criterion.

The following three elements constitute the optimal control formulation|:

- model (a mathematical description) of the process/system to be controlled
- mathematical description of the (physical) constraints of the system
- a performance measure and its mathematical description

- **Minimum-time problem**: To transfer a system from arbitrary initial state $x(t_0) = x_0$ to a specified target set $\mathcal{S}$ in minimum time

$$J = t_f - t_0 = \int_{t_0}^{t_f} dt, \tag{1}$$

where t_f is the first instant of time when $\mathbf{x}(t)$ and $\mathcal{S}$ intersect.

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For discrete-time systems, minimum-time performance can be cast as

$$J = N = \sum_{k=0}^{N-1} 1.$$

Performance measures

- **Terminal control problem**: to minimize the deviation of the final state of a system from its desired value $r(t_f) \in \mathbb{R}^n$

$$J = \sum_{i=1}^{n} (x_i(t_f) - r_i(t_f))^2 = (x(t_f) - r(t_f))^\top (x(t_f) - r(t_f)) = \|x(t_f) - r(t_f)\|^2.$$

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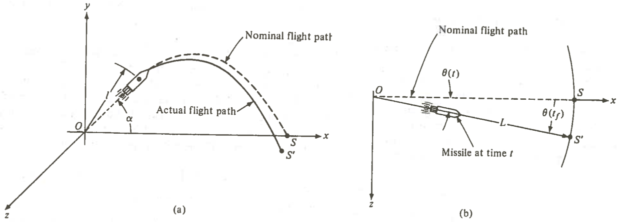

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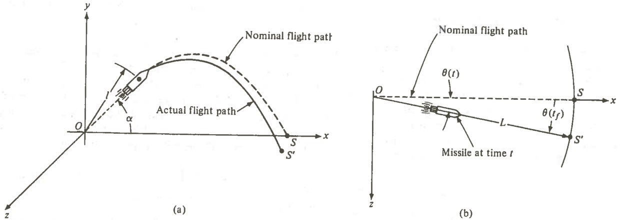

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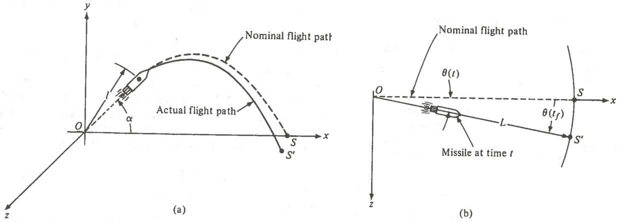

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    $$J = \frac{1}{2} \sum_{k=0}^{N-1} u_k^\top R u_k.$$

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- **Regulation problem**: $r(t) = 0$ for all $t \in [t_0, t_f]$

Performance measures

All the performance measures discussed above are special cases of the general form

- Continuous-time

$$J = \underbrace{h(x(t_f), t_f)}_{\text{terminal cost}} + \underbrace{\int_{t_0}^{t_f} g(x(t), u(t), t) dt}_{\text{running cost}}.$$

- Discrete-time

$$J = \underbrace{\phi(x_N, N)}_{\text{terminal cost}} + \underbrace{\sum_{k=0}^{N-1} L^k(x_k, u_k) dt}_{\text{running cost}}.$$

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  - pro: choose among multiple possibilities accounting for other measures

Parameter static optimization: when time is not a parameter in the problem

- Unconstrained optimization
- Constrained optimization

Unconstrained optimization

$$u^\star = \underset{u \in \mathbb{R}^m}{\operatorname{argmin}}\ F(u),$$

where $F : \mathbb{R}^m \to \mathbb{R}$ is differentiable

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Local (weak) minimum point: A point $u^\star \in \mathbb{R}^m$ is said to be a *(weak) local minimum point* of F over $\mathbb{R}^m$ if

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Local (strong) minimum point:

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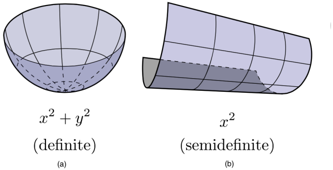

(a) strong minimum, (b) weak minimum

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$$u^\star = \underset{u \in \mathbb{R}^m}{\text{argmin}}\ F(u),$$

where $F : \mathbb{R}^m \to \mathbb{R}$ is differentiable

A point $u^\star \in \mathbb{R}^m$ is said to be a **Local (weak) minimum point** of F over $\mathbb{R}^m$ if

$$\exists \epsilon > 0 \text{ s.t. } F(u^\star) \leqslant F(u)\ \forall u \in \mathbb{R}^m, \quad \|u - u^\star\| < \epsilon$$

Local (strong) minimum point: $\exists \epsilon > 0 \text{ s.t. } F(u^\star) < F(u)\ \forall u \in \mathbb{R}^m, \quad \|u - u^\star\| < \epsilon$

Global minimum: (weak) $F(u^\star) \leqslant F(u)\ \forall, u \in \mathbb{R}^m$, (strong) $F(u^\star) < F(u)\ \forall, u \in \mathbb{R}^m$

Unconstrained optimization

$$u^\star = \underset{u \in \mathbb{R}^m}{\text{argmin}}\ F(u),$$

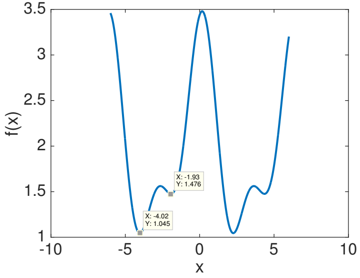
where $F : \mathbb{R}^m \to \mathbb{R}$ is differentiable

A point $u^\star \in \mathbb{R}^m$ is said to be a **Local (weak) minimum point** of F over $\mathbb{R}^m$ if

$$\exists \epsilon > 0 \text{ s.t. } F(u^\star) \leqslant F(u)\ \forall u \in \mathbb{R}^m, \quad \|u - u^\star\| < \epsilon$$

Local (strong) minimum point: $\exists \epsilon > 0 \text{ s.t. } F(u^\star) < F(u)\ \forall u \in \mathbb{R}^m, \quad \|u - u^\star\| < \epsilon$

Global minimum: (weak) $F(u^\star) \leqslant F(u)\ \forall, u \in \mathbb{R}^m$, (strong) $F(u^\star) < F(u)\ \forall, u \in \mathbb{R}^m$



$f(x) = 2 + \cos(x) + 0.5\ \cos(2x - 0.5)$ has multiple local and global minimizer.