Homework Assignment 2
Optimal Control- MAE 274
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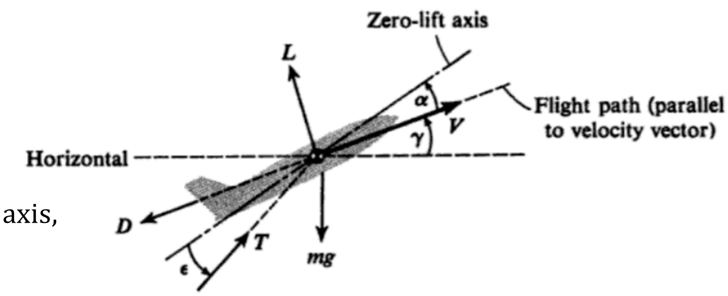
Turn in your HW and electronically to the respective folder in Canvas. Include the codes with your written HW (no need to turn in your Matlab m codes separately).

Problem 1 (Bryson and Ho). In this problem we want to find control inputs that maximize steady rate of climb for an aircraft. The net force on an aircraft maintaining a steady state rate of climb must be zero.

V= velocity,
γ= flight path angle to horizontal,
α=angle-of-attack,
m=mass of aircraft,
g=gravitational force per unit mass,
ϵ=angle between thrust axis and zero lift axis,
At the given altitude,
L=L(V,α)=lift force,
D=D(V,α)=drag force,
T=T(V)=thrust force of engine,
$V\sin(\gamma)$ =rate of climb.



We choose V and γ as state parameters and α as the control parameter since, at a given altitude, a choice of α determines V and γ from the force equilibrium relations.

(a) Use the force diagram to write the force equilibrium equations in the parallel and perpendicular directions to the flight path angle.
(b) Set up the optimization problem for maximizing the rate of the climb (you can minimize the negative of your objective function to set you problem as minimization problem). For this part you need to specify your cost function and the equality constraints that relate the state and control variables.
(c) Use the method of multipliers and obtain the first order conditions to find stationary points of your optimal cost on the constraint manifold (you do not need to solve the equations. Only identify the equations and unknowns you have).

Problem 2.
(a) Find the scalar quantity that minimizes

$$F(x,u) = \frac{1}{2}\left(\frac{x^2}{a^2} + \frac{u^2}{b^2}\right)$$

subject to the quadratic constraint

$$f(x,u) = c - xu = 0.$$

Here $a, b, c \in R_{>0}$ are unknown constants. What is the value of your optimal cost $F(x^*, u^*)$? Evaluate the second order sufficiency condition and determine if your critical point is a minimum point.

Let $a = 2, b = 1, c = 2$.

(b) Solve the problem using Matlab fmincon.

(c) Draw the constraint curve and $\frac{1}{2}\left(\frac{x^2}{a^2} + \frac{u^2}{b^2}\right) = l$ for $l = \{F(x^*, u^*), F(x^*, u^*) + 1, F(x^*, u^*) + 0.5, F(x^*, u^*) - 0.5\}$. What observation can you make from this figure about the optimum point (x^*, u^*)?

Problem 3. Do problem 2.1.2 from Ref. [2]. This reference is available at UCI library as an e-book. Also you can obtain a copy from the link below
http://www.uta.edu/utari/acs/FL%20books/Lewis%20optimal%20control%203rd%20edition%2 02012.pdf

Problem 4. Consider the scalar plant

$$x_{k+1} = 2x_k + u_k.$$

Starting from $x_0 = 3$, our goal is to find controller sequences that minimize

$$J = \frac{1}{2}\sum_{k=0}^{3} u_k^2$$

and take the plant to $x_4 = 5$.

(a) Write down the stationary point equations (state, co-state etc) and determine the optimum control sequence using these equations.

(b) Cast this problem as a static constrained optimization problem and solve this problem using Matlab fmincon solver.

(c) Write down the state equations for $k = 0,1,2,3,4$. Substitute for initial state as $x_0 = 3$ and $x_4 = 5$. Cast the state equations as linear algebra equation $Az = b$. Characterize all the solutions z (using null-space and particular solution).

(d) Choose two feasible solutions for control sequence from part (c) and evaluate the cost function for them. Compare the values you get with the optimum value from the earlier part. Plot the state trajectories and control inputs for these three set of control sequences you have (a plot for x(k) vs k and a plot for u(k) vs. k).

Problem 5. Read the paper referenced below and write half a page about it

Arthur Bryson, ``Optimal control- 1950-1985”.

In particular, identify some of the milestones and key contributors/contributions discussed in this paper. The paper is available in class website.

Matrix differentiation:
The first order necessary conditions in Ref[2] is derived based on column representation of gradient, i.e, for a Function $F(x): R^n \to R$ we have

$$\frac{\partial F(x)}{\partial x} = \begin{bmatrix} \frac{\partial F(x)}{\partial x_1} \\ \vdots \\ \frac{\partial F(x)}{\partial x_n} \end{bmatrix}$$

Keeping this in mind, we have the following formula's for quadratic term differentiation

$$\frac{\partial (x^T Q x)}{\partial x} = (Q + Q^T)x$$

$$\frac{\partial (x^T y)}{\partial x} = y$$

$$\frac{\partial (x^T Q y)}{\partial x} = Qy$$

$$\frac{\partial (x^T Q y)}{\partial y} = Q^T y$$

$$\frac{\partial (x^T y)}{\partial y} = x$$

See the following examples

$$x^T Q x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = q_{11} x_1^2 + q_{12} x_1 x_2 + q_{21} x_2 x_1 + q_{22} x_2^2$$

$$\frac{\partial (x^T Q x)}{\partial x} = \begin{bmatrix} \frac{\partial (x^T Q x)}{\partial x_1} \\ \frac{\partial (x^T Q x)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2q_{11} x_1 + q_{12}\, x_2 + q_{21} x_2 \\ 2q_{22} x_2 + q_{12}\, x_1 + q_{21} x_1 \end{bmatrix} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$
$$= (Q + Q^T)x$$

or

$$x^T Q y = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = q_{11} x_1 y_1 + q_{12} x_1 y_2 + q_{21} x_2 y_1 + q_{22} x_2 y_2$$

$$\frac{\partial (x^T Q y)}{\partial x} = \begin{bmatrix} \frac{\partial (x^T Q y)}{\partial x_1} \\ \frac{\partial (x^T Q y)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} q_{11} y_1 + q_{12}\, y_2 \\ q_{22} y_2 + q_{21} y_1 \end{bmatrix} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Qy$$

$$\frac{\partial (x^T Q y)}{\partial y} = \begin{bmatrix} \frac{\partial (x^T Q y)}{\partial y_1} \\ \frac{\partial (x^T Q y)}{\partial y_2} \end{bmatrix} = \begin{bmatrix} q_{11} x_1 + q_{21}\, x_2 \\ q_{22} x_2 + q_{12} x_1 \end{bmatrix} = \begin{bmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Q^T x$$