MAE270A: Concepts of Realization Theory
Minimal Realization for LTI systems
Some of the properties of minimal realization of a system
Instructor: Solmaz Kia (solmaz@uci.edu), University of California Irvine

Problem setting.

An LTI system can be represented by

- a transfer function $G(s)$ in frequency domain
- a linear differential equation below in time domain

$$(A,B,C,D) \sim \begin{cases} \dot{x} = Ax + Bu & x \in \mathbb{R}^n,\ u \in \mathbb{R}^p \\ y = Cx + Du & y \in \mathbb{R}^q \end{cases} \tag{1}$$

Research question.

How are $G(s)$ and (A,B,C,D) related?

From (A,B,C,D) to $G(s)$.

$G(s)$ explains the input to out relation for a relaxed system (zero initial conditions):

$$Y(s) = G(s)U(s), \quad \text{where} \quad Y(s) = \mathcal{L}[y(t)], \quad U(s) = \mathcal{L}[u(t)].$$

Transfer function of an LTI system (1) is obtained as follows:

$$\mathcal{L}[\dot{x}] = L[Ax + Bu] \rightarrow sX(s) - x(0) = AX(s) + BU(s)$$

$$\rightarrow X(s) = (sI - A)^{-1}BU(s).$$

$$\downarrow$$

$$\mathcal{L}[y] = \mathcal{L}[Cx + Du] \rightarrow Y(s) = \underbrace{\left(C(sI - A)^{-1}B + D\right)}_{G(s)} U(s)$$

$$\downarrow$$

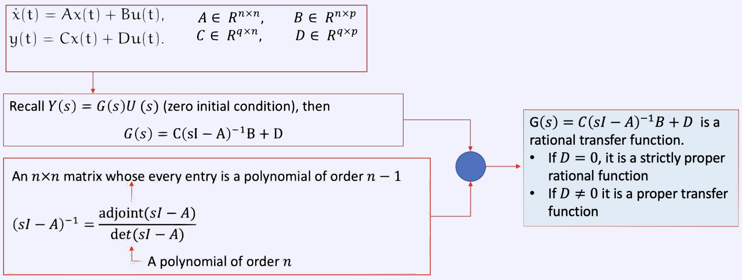
$$G(s) = C(sI - A)^{-1}B + D \tag{2}$$

Note that

- $(sI - A)^{-1} = \frac{\text{adj}(sI-A)}{\det(sI-A)}$
- $\det(sI - A) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n$
- Every entry (i,j) of $\text{adj}(sI - A)$ is obtained from $(-1)^{(i+j)}$ times the determinant of the $(n-1) \times (n-1)$ submatrix of $(sI - A)$ where the column i and row j of $(sI - A)$ is dropped.

Consequently, every entry of $G(s)$ matrix is a proper rational transfer function, i.e., every entry is a ratio of two polynomial where the degree of denominator is greater than or equal to the degree of the numerator polynomial.

From (A, B, C, D) to $G(s)$, cont'd.



Definition:
Zero-state equivalence.

Two state-space systems (A, B, C, D) and $\bar{A}, \bar{B}, \bar{C}, \bar{D})$ are said to be <u>zero-state equivalent</u> if they have the same transfer function, i.e., $G(s) = C(sI - A)^{-1}B + D = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D}$.

Zero-state equivalent systems exhibit the same zero-state response when exposed to a same input.

Zero-equivalent systems does not necessarily have the same state dimension.

The following systems are zero-state equivalent

$$A = \left[\begin{array}{cc|cc|cc} -4.5 & 0 & -6 & 0 & -2 & 0 \\ 0 & -4.5 & 0 & -6 & 0 & -2 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right], \quad B = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right]$$

$$C = \left[\begin{array}{cc|cc|cc} -6 & 3 & -24 & 7.5 & -24 & 3 \\ 0 & 1 & 0.5 & 1.5 & 1 & 0.5 \end{array}\right], \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} -2.5 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -4 & -4 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} -6 & -12 & 3 & 6 \\ 0 & 0.5 & 1 & 1 \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$G(s) = C(sI - A)^{-1}B + D = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D}$$

$$G(s) = \begin{bmatrix} \frac{2(s-2.5)}{s+0.5} & \frac{3}{s+2} \\ \frac{0.5}{(s+2)(s+0.5)} & \frac{s+1}{(s+2)^2} \end{bmatrix}$$

Definition:
algebraically equivalence.

Consider

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$

Given T <u>nonsingular</u>, apply change of variable $\bar{x} = Tx$ to write the system in the new state $\bar{x}$

$$\begin{cases} \dot{\bar{x}} = T\dot{x} = T(Ax(t) + Bu(t)) = \underbrace{TAT^{-1}}_{\bar{A}}\bar{x} + \underbrace{TB}_{\bar{B}}u(t) \\ y(t) = C(t)x(t) + D(t)u(t) = \underbrace{CT^{-1}}_{\bar{C}}\bar{x} + \underbrace{D}_{\bar{D}}u(t) \end{cases} \Rightarrow \begin{cases} \dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t), \\ y(t) = \bar{C}\bar{x}(t) + \bar{D}u(t), \end{cases}$$

Def(Algebraically equivalent) Two continuous-time LTI systems

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \quad \text{and} \quad \begin{cases} \dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t), \\ y(t) = \bar{C}\bar{x}(t) + \bar{D}u(t), \end{cases}$$

are called algebraically equivalent if and only if there exists a nonsingular T s. t. ($\bar{A} = TAT^{-1}$, $\bar{B} = TB$, $\bar{C} = CT^{-1}$, $\bar{D} = D$). The corresponding map $\bar{x} = Tx$ is called a <u>similarity transformation</u> or an <u>equivalence transformation</u>.

Some properties of algebraically equivalent systems.

P1. With every input signal u, both systems associate the same set of outputs y. However, the output is generally not the same for the same initial conditions, except for the forced or zero-state response, which is always the same.

P2. the systems are zero-state equivalent, i.e., they have the same transfer function.

$$\bar{C}(sI-\bar{A})^{-1}\bar{B}+\bar{D}=C(sI-A)^{-1}B+D$$

$$\begin{aligned}\bar{C}(sI-\bar{A})^{-1}\bar{B}+\bar{D}=&CT^{-1}(sI-TAT^{-1})^{-1}TB+D=\\&CT^{-1}(sTT^{-1}-TAT^{-1})^{-1}TB+D=\\&CT^{-1}(T(sI-A)^{-1}T^{-1})TB+D=\\&C(sI-A)^{-1}B+D.\end{aligned}$$

Attention: In general the converse of P2. does not hold, i.e., zero-state equivalence does not imply algebraic equivalence. For two state equations to be equivalent, they must have the same dimension. This is, however, is not required for zero-state equivalent systems.

P3. they have the same eigenvalues.[1]

$$\bar{\Delta}(\lambda)=\det(\lambda I-\bar{A})=\det(\lambda I-A)=\Delta(\lambda)$$

The equivalent state equations have the same characteristic polynomial and consequently the same se of eigenvalues.

$$\begin{aligned}\bar{\Delta}(\lambda)=\det(\lambda I-\bar{A})=\det(\lambda TT^{-1}-TAT^{-1})=\det(T)\det(\lambda I-A)\det(T^{-1})=\\=\det(\lambda I-A)\det(T)\det(T^{-1})=\det(\lambda I-A)=\Delta(\lambda).\end{aligned}$$

[1] recall $\det(AB)=\det(A)\det(B)=\det(B)\det(A)$

Realization theory: from a proper rational $G(s)$ to (A,B,C,D).

Realizable Transfer Function A transfer function $G(s)$ is said to be realizable if there is a finite dimensional state-space equation (1) or simply (A,B,C,D) such that

$$G(s)=C(sI-A)^{-1}B+D.$$

Then, (A,B,C,D) is called a realization of $G(s)$.

Note every $G(s)$ is realizable as (A,B,C,D). Recall that distributed systems have an impulse response and, as a result, a transfer function, but they do not have a state-space realization in the form of (1). An example of such distributed systems is $G(s)=\frac{e^{-s}}{s+1}$.

Theorem (realizable transfer function): A transfer function $G(s)$ can be realized as a state-space equation (1) if and only if $G(s)$ is a proper rational transfer function.

If a transfer function $G(s)$ is realizable, it can have an infinite number of realizations that are not necessarily of the same order. All of these realizations will have the same number of inputs and the same number of outputs, but the order of the system matrix $A\in\mathbb{R}^{n\times n}$ may differ between them.

The problem of interest now is how to obtain a realization (A,B,C,D) from a given transfer function $G(s)$, i.e., finding (A,B,C,D) that satisfies $G(s)=C(sI-A)^{-1}B+D$.

Next, we demonstrate the procedure to solve this problem using a running example.

The procedure to obtain a realization (A, B, C, D) from a given $G(s)$.

We demonstrate the step for

$$G(s) = \begin{bmatrix} \frac{s+1}{s+3} \\ \frac{s-1}{s+1} \\ \frac{s+2}{(s+1)(s+3)} \end{bmatrix}$$

number of input: $p = 1$; number of outputs $q = 3$.

- Write $G(s) = G_{sp}(s) + D$, where $D = \lim_{s\to\infty} G(s)$

$D = \lim_{s\to\infty} G(s) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $G_{sp}(s) = G(s) - D =$

$$\begin{bmatrix} \frac{s+1}{s+3} \\ \frac{s-1}{s+1} \\ \frac{s+2}{(s+1)(s+3)} \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{2}{s+3} \\ -\frac{2}{s+1} \\ \frac{s+2}{(s+1)(s+3)} \end{bmatrix}$$

- Find the monic least comment denominator of all the entries of $\hat{G}_{sp}(s)$ matrix,

$$d(s) = 1s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n.$$

The monic least common denominator of a family of polynomials is the monic polynomial of the smallest order that can be divided by all those polynomials.

$$d(s) = (s+1)(s+3) = 1\,s^2 + \underbrace{4}_{\alpha_1}\,s + \underbrace{3}_{\alpha_2}$$

- $G_{sp}(s) = \frac{1}{d(s)}[N_1 s^{n-1} + N_2 s^{n-2} + \cdots + N_{n-1}s + N_n]$,

$$\begin{bmatrix} -\frac{2}{s+3} \\ -\frac{2}{s+1} \\ \frac{s+2}{(s+1)(s+3)} \end{bmatrix} = \frac{1}{s^2+4s+3}\begin{bmatrix} -2(s+1) \\ -2(s+3) \\ s+2 \end{bmatrix} = \frac{1}{s^2+4s+3}\left(\underbrace{\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}}_{N_1} s + \underbrace{\begin{bmatrix} -2 \\ -6 \\ 2 \end{bmatrix}}_{N_2}\right)$$

- We claim (controllable canonical form)

p: number of inputs
q: number of outputs

$$A = \begin{bmatrix} -\alpha_1 I_{p\times p} & -\alpha_2 I_{p\times p} & \cdots & -\alpha_{n-1} I_{p\times p} & -\alpha_n I_{p\times p} \\ I_{p\times p} & 0_{p\times p} & \cdots & 0_{p\times p} & 0_{p\times p} \\ 0_{p\times p} & I_{p\times p} & \cdots & 0_{p\times p} & 0_{p\times p} \\ \vdots & & \ddots & \vdots & \vdots \\ 0_{p\times p} & 0_{p\times p} & \cdots & I_{p\times p} & 0_{p\times p} \end{bmatrix}_{np\times np}, \quad B = \begin{bmatrix} I_{p\times p} \\ 0_{p\times p} \\ \vdots \\ 0_{p\times p} \\ 0_{p\times p} \end{bmatrix}_{np\times p}$$

$$C = \begin{bmatrix} N_1 & N_2 & \cdots & N_{n-1} & N_n \end{bmatrix}_{q\times np}, \quad D = \lim_{s\to\infty} \hat{G}(s)$$

$$A = \begin{bmatrix} -4 & -3 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} -2 & -2 \\ -2 & -6 \\ 1 & 2 \end{bmatrix}, D = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

We can show through direct substitution and some algebraic manipulations that the claimed (A, B, C, D) matrices in the previous step satisfy

$$G_{sp}(s) = C(sI - A)^{-1}B.$$

Note that the procedure outlined above is one of the methods to obtain a realization of transfer function. In a similar approach we can obtain the observable canonical realization of $G(s)$.

Exercise.

Follow the procedure outlined above to arrive at the realization given below:

$$\hat{G}(s) = \begin{bmatrix} \frac{s}{(2s+1)} & \frac{s+1}{(s+2)} & \frac{1}{(s+1)} \\ \frac{s+3}{(s+1)} & \frac{s^2}{(s+\frac{1}{2})(2s+1)} & \frac{-1}{(4s+8)} \end{bmatrix}$$

$$\Downarrow$$

$$A = \left[\begin{array}{ccc|ccc|ccc|ccc} -4 & 0 & 0 & -\frac{21}{4} & 0 & 0 & -\frac{11}{4} & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & -4 & 0 & 0 & -\frac{21}{4} & 0 & 0 & \frac{11}{4} & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -4 & 0 & 0 & -\frac{21}{4} & 0 & 0 & \frac{11}{4} & 0 & 0 & -\frac{1}{2} \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right], \quad B = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$C = \left[\begin{array}{ccc|ccc|ccc|ccc} \frac{-1}{4} & -1 & 1 & \frac{-7}{8} & -2 & 3 & \frac{-7}{8} & -\frac{5}{4} & \frac{9}{4} & \frac{-1}{4} & -\frac{1}{4} & \frac{1}{2} \\ 2 & \frac{-1}{2} & \frac{-1}{4} & 6 & \frac{-13}{8} & \frac{-7}{8} & \frac{9}{2} & \frac{-11}{8} & \frac{-7}{8} & 1 & \frac{-1}{4} & \frac{-1}{4} \end{array}\right], \quad D = \begin{bmatrix} \frac{1}{2} & 1 & 0 \\ 1 & \frac{1}{2} & 0 \end{bmatrix}$$

Research question.

We saw in the derivation and the discussion so far that the realization of transfer function is not unique and they do not even necessarily have same dimension. So the question is

- What is the minimal realization of transfer function?
- How to obtain the minimal realization of a given transfer function?
- Is the minimal realization of a transfer function unique?

Motivating example.

Earlier in this note, we saw two different realizations for the transfer function

$$G(s) = \begin{bmatrix} \frac{2(s-2.5)}{s+0.5} & \frac{3}{s+2} \\ \frac{0.5}{(S+2)(s+0.5)} & \frac{s+1}{(s+2)^2} \end{bmatrix}. \tag{3}$$

One of there realizations was of order 6 and the the other was of order 4. Interestingly, even the realization with order 4 is not the minimal realization for this given $G(s)$.

Definition: Minimal Realization.

A realization of transfer function $G(s)$ is <u>minimal</u> or irreducible if there is no other realization of $G(s)$ of smaller order.

Minimal Realization: main theorem.

A realization (A, B, C, D) is minimal if and only if it is both controllable and observable.

The proof is related to the Kalman decomposition of a realization. Recall that after Kalman decomposition identifies the part of the system that is both controllable and observable, $(A_{co}, B_{co}, C_{co}, D)$, we have

$$G(s) = C(sI - A)^{-1}B + D = C_{co}(sI - A_{co})^{-1}B_{co} + D$$

Back to motivating example.

Now let us check why the 4th order realization

$$A = \begin{bmatrix} -2.5 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -4 & -4 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} -6 & -12 & 3 & 6 \\ 0 & 0.5 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

of the transfer function (3) is not a minimal realization. The answer comes from checking whether this system is controllable or observable. If we are told that this realization is not the minimal realization of $G(s)$, it means that this system lacks at least one of the controllability and observability properties. For this system we have

$$\text{rank}(\mathcal{C}) = 4, \qquad \text{rank}(\mathcal{O}) = 3.$$

As we can see this realization is not observable and thus we can conclude that this realization is not minimal. We can expect that after removing the observable part, the third order realization we obtain is minimal.

The observable decomposition for this 4th order system is

$$A = \left[\begin{array}{ccc|c} -0.8258 & 3.2845 & -2.4395 & 0 \\ -0.3119 & -2.8714 & 1.5096 & 0 \\ 0.2022 & 0.5678 & -0.8028 & 0 \\ \hline 0.9417 & 2.6377 & 0.5965 & -2 \end{array}\right], \quad B = \left[\begin{array}{cc} -0.4440 & 0.6189 \\ -0.0543 & -0.6455 \\ -0.3999 & 0.2009 \\ \hline 0.8000 & 0.4000 \end{array}\right]$$

$$C = \left[\begin{array}{ccc|c} 0 & 0.0200 & 15.0000 & 0 \\ 0 & -1.4863 & 0.2020 & 0 \end{array}\right], \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, the observable part is

$$A_o = \begin{bmatrix} -0.8258 & 3.2845 & -2.4395 \\ -0.3119 & -2.8714 & 1.5096 \\ 0.2022 & 0.5678 & -0.8028 \end{bmatrix}, \quad B_o = \begin{bmatrix} -0.4440 & 0.6189 \\ -0.0543 & -0.6455 \\ -0.3999 & 0.2009 \end{bmatrix}$$

$$C_o = \begin{bmatrix} 0 & 0.0200 & 15.0000 \\ 0 & -1.4863 & 0.2020 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

Note: due to rounding error, the transfer function obtained via Matlab may not exactly look like $G(s)$. After proper rounding (A_o, B_o, C_o, D) realizes the same $G(s)$ in (3), as expected. (A_o, B_o) is controllable. Thus, we can conclude that (A_o, B_o, C_o, D) is a minimal realization of $G(s)$ in (3)

Some of the properties of minimal realizations.

Theorem: All minimal realization of a transfer function are algebraically equivalent.

Theorem: Consider a SISO system with transfer function $g(s) = \frac{n(s)}{d(s)}$, where $d(s)$ is a monic polynomial and $d(s)$ and $n(s)$ coprime. In this case, the realization (A, B, C, D) where $A \in \mathbb{R}^n$ is minimal if and only if n is equal to the degree of $d(s)$. In this case $d(s)$ is equal to the characteristic polynomial of A, i.e., $d(s) = \det(sI - A)$

Theorem: Let (A, B, C, D) be a minimal realization of a SISO transfer function $g(s)$. Then, $g(s)$ is BIBO stable if and only if the realization is internally asymptotically stable.

Practice example.

Q: Consider the realization (A, B, C, D) given by

$$A = \begin{bmatrix} -2 & 0 \\ a & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = 0.$$

For what values of a, this system is a minimal realization?

A: For a realization to be minimal, it needs to be both controllable and observable:

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & a+1 \end{bmatrix} \rightarrow \text{rank}(\mathcal{C}) = 2, \quad \forall a \in \mathbb{R} \backslash \{-3\}$$

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a & 1 \end{bmatrix} \rightarrow \text{rank}(\mathcal{O}) = 2, \quad \forall a \in \mathbb{R} \backslash \{0\}.$$

Therefore, (A, B, C, D) is a minimal realization for any $a \in \mathbb{R} \backslash \{0, -3\}$.

We can also obtain the same result by examining the transfer function of the system:

$$G(s) = C(sI - A) - 1B + D = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+2 & 0 \\ -a & s-1 \end{bmatrix} \Big) ^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{s+2+a}{(s+2)(s-1)} = \begin{cases} \frac{1}{(s-1)} & a = 0 \\ \frac{1}{s+2} & a = -3 \end{cases}$$

For the realization (A, B, C, D) to be minimal, the charactristic polynomial of the transfer function $G(s)$ should be of order two. Therefore, any value of a that results in pole-zero cancellation should be avoided. Consequently, (A, B, C, D) is a minimal realization for any $a \in \mathbb{R} \backslash \{0, -3\}$.

Appendix: some definitions about the transfer functions.

For a SISO system:

- Rational Transfer Function is a ratio of two polynomials with real coefficients:
  Q: Which of the transfer functions below is the rational transfer function?

$$(a)\ \ g(s)=\frac{s^2+1}{s^2+2s+1},\quad (b)\ \ g(s)=\frac{s^2+1}{s^2+2s+1}\mathrm{e}^{-0.2s}.$$

  A: (a)

- Proper Rational Transfer Function is a transfer function in which the degree of the numerator does not exceed the degree of the denominator, otherwise the transfer function is Improper.
  Q: Which of the transfer functions below is a proper rational transfer function?

$$\text{(a)}\ \ g(s)=\frac{s^3+1}{s^2+2s+1},\quad \text{(b)}\ \ g(s)=\frac{s^2+1}{s^2+2s+1}.$$

  A: (b)

- Strictly Proper Rational Transfer Function is a transfer function in which the degree of the numerator is less than the degree of the denominator. Q: Which one of the transfer functions below is strictly proper rational transfer function?

$$\text{(a)}\ \ g(s)=\frac{s+1}{s^2+2s+1},\quad \text{(b)}\ \ g(s)=\frac{s^2+1}{s^2+2s+1}.$$

  A: (a)

For an MIMO system, with p inputs and q outputs, the transfer function is a $q\times p$ matrix

$$G(s)=\begin{bmatrix} g_{11}(s) & g_{12}(s) & \cdots & g_{1p}(s)\\ \vdots & \vdots & \dots & \vdots\\ g_{q1}(s) & g_{q2}(s) & \cdots & g_{qp}(s)\end{bmatrix},$$

where each entry (i,j) is the transfer function relationship between input j and output i.

- $G(s)$ is rational transfer function when all entry g_{ij}s are rational transfer functions.
- $G(s)$ is proper rational transfer function when all entry $g_{ij}(s)$'s are proper rational transfer functions. Moreover, $G(s)$ is strictly proper if all entries are strictly proper.

References.

- Joao P. Hespanha. Linear Systems Theory, 2009
- C. T. Chen. Linear System Theory and Design, 4th edition.
- Franklin and Powell. Feedback Control