Linear Systems I
Lecture 3

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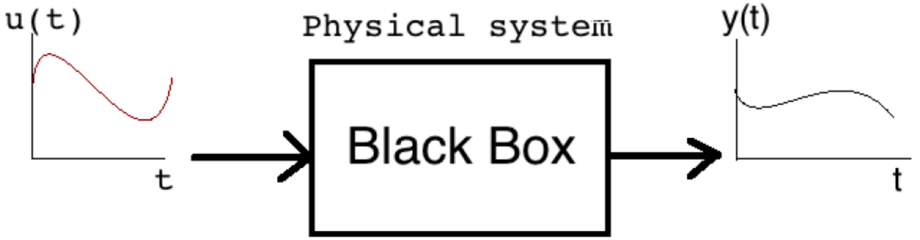
Outline

Linear system

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t).$$



- Basic properties of LTV/LTI systems
  - Causality
  - Linearity
  - Time invariance

Input-output description (Impulse response) for relaxed and linear system

$$y(t) = \int_{t_0}^{t} G(t,\tau)\, u(\tau) d\,\tau, \qquad \forall t \geq 0$$

$G(t,\tau)$ is the system's output at time t due to an impulse at time τ.

Input-output description (Impulse response) for relaxed and linear time-invariant system

$$y(t) = \int_{t_0}^{t} G(t)\, u(t-\tau) d\,\tau, \qquad \forall t \geq 0$$

$G(t)$ is the system's output at time t due to an impulse at time 0.

Theorem: Let y be an output corresponding to a given input u of a linear system. All outputs corresponding to u can be obtained by

Response=zero-input response + zero-state response

$$y = y_{zs} + y_{zi}$$

To construct all the outputs due to u:

- Find one particular output corresponding to the input u and zero initial condition.
- Final all outputs corresponding to the zero input.

Input-output description in Laplace domain

$$\hat{y}(s) = \hat{G}(s)\,\hat{u}(s), \qquad \forall t \geq 0$$

Lecture 3 covers

- Review of Impulse Response and Transfer Function
- From LTI State-Space form to Transfer Function
- From Transfer Function to LTI State-Space form

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Basic properties of LTV/LTI systems: Causality

Every linear, time-invariant system has a transfer function.

$$\hat{G}(s) = \mathcal{L}[G(t)] = \int_0^\infty G(t)e^{-st}\,dt, \quad s \in \mathbb{C},$$

relates the zero state response of a system to its input ($\hat{y}(s) = \hat{G}(s)\hat{u}(s)$, $x(0) = 0$).

Every linear, time-invariant, lumped system can be represented by a state-space representation

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du$$

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From SS representation to TF representation

State-space (SS) representation of a linear, time-invariant and lumped system

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t).$$

Theorem: The impulse response and transfer function of an LTI system represented by the SS form above are given by, respectively,

$$\hat{G}(s) = C(sI - A)^{-1}B + D, \quad G(t) = \mathcal{L}^{-1}[C(sI - A)^{-1}B + D].$$

Moreover, the output given by

$$y(t) = \int_0^t G(t-\tau)u(\tau)d\tau = \int_0^t G(\tau)u(t-\tau)d\tau$$

corresponds to the zero initial condition $x(0) = 0$ solution of the SS form above.

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From SS representation to TF representation

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t).$$

$$\mathcal{L}[\dot{x}(t)] = s\hat{x}(s) - x(0),$$

$$s\hat{x}(s) - x(0) = A\hat{x}(s) + B\hat{u}(s)$$
$$\hat{y}(s) = C\hat{x}(s) + D\hat{u}(s)$$

$$\hat{x}(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}B\hat{u}(s),$$
$$\hat{y}(s) = \underbrace{C(sI - A)^{-1}x(0)}_{\text{response due to initial condition}} + \underbrace{(C(sI - A)^{-1}B + D)\hat{u}(s)}_{\text{response due to input}} \quad (\star).$$

Recall $\hat{y}(s) = \hat{G}(s)\hat{u}(s)$ (zero initial condition), then

$$\hat{G}(s) = C(sI - A)^{-1}B + D$$

Take inverse Laplace

$$y(t) = \psi(t)x(0) + \int_0^t G(t-\tau)u(\tau)d\tau,$$

$$G(t) = \mathcal{L}^{-1}[\hat{G}(s)], \quad \psi(t) = \mathcal{L}^{-1}[\hat{\psi}(s)].$$

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From SS representation to TF representation

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t).$$

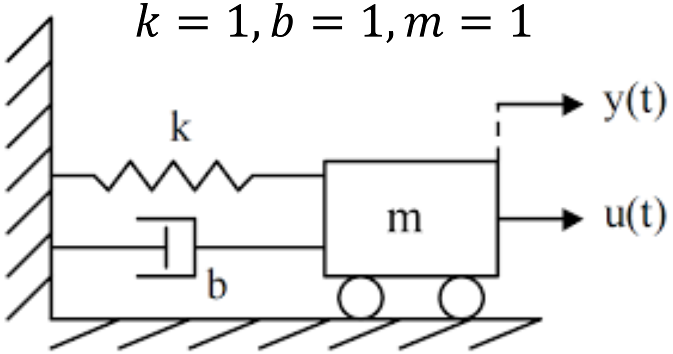
Recall $\hat{y}(s) = \hat{G}(s)\hat{u}(s)$ (zero initial condition), then

$$\hat{G}(s) = C(sI - A)^{-1}B + D$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{B} u$$

$$y = d = \underbrace{[1 \quad 0]}_{C}\, x + \underbrace{0}_{D}\, u$$

$k = 1, b = 1, m = 1$



$$\hat{G}(s) = C(sI - A)^{-1}B + D$$

$$\hat{G}(s) = [1 \quad 0]\left(s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{s^2+s+1}$$

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Elementary Realization (from TF rep. to SS rep.)

Def. (Realization problem): how to compute SS representation from a given transfer function.

- Note every TF is not necessarily realizable. Recall that distributed systems have impulse response and as a result transfer function but no SS rep.

Def. (Realizable TF): A transfer function $\hat{G}(s)$ is said to be realizable if there exists a finite dimensional SS equation

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$

or simply $\{A, B, C, D\}$ such that

$$\hat{G}(s) = C(sI - A)^{-1}B + D.$$

We call $\{A, B, C, D\}$ a realization of $\hat{G}(s)$.

Note: if a transfer function is realizable it has infinitely many realization, not necessarily of the same dimension.

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Realizable transfer functions

Rational Transfer Function is a ratio of two polynomials with real coefficients.

$$\hat{G}(s) = \frac{s^2 + 1}{s^3 + s + 1} \qquad \hat{G}(s) = \frac{s^2}{s^3 + s + 1} e^{-0.5s}$$

Proper Rational Transfer Function is a transfer function in which the degree of the numerator does not exceed the degree of the denominator, otherwise the transfer function is **Improper**

$$\hat{G}(s) = \frac{3+1}{s^3+s+1} \qquad \hat{G}(s) = \frac{s^4+1}{s^3+s+1}$$

Strictly Proper Rational Transfer Function is a transfer function in which the degree of the numerator is less than the degree of the denominator,

$$\hat{G}(s) = \frac{s^2+1}{s^3+s+1} \qquad \hat{G}(s) = \frac{s^3+1}{s^3+s+1}$$

Theorem (realizable transfer function: A transfer function $\hat{G}(s)$ can be realized by an LTI SS equation iff ($\Leftrightarrow$) $\hat{G}(s)$ is a proper rational function.

Proof:

- ($\Rightarrow$) from a SS realization to a proper rational TF
- ($\Leftarrow$) from a proper rational TF to a SS realization

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From rational proper TF to SS

Consider a linear causal system with p inputs and q outputs.

a proper TF $\hat{G}(s) \Rightarrow$ **SS representation**:
find $\{A, B, C, D\}$ such that $\hat{G}(s) = C(sI - A)^{-1}B + D$.

$$\hat{G}(s) = \begin{bmatrix} \frac{s+1}{s+3} \\ \frac{s-1}{s+1} \\ \frac{s+2}{(s+1)(s+3)} \end{bmatrix},$$

$p = 1$ input and $q = 3$ output.

- Write $\hat{G}(s) = \hat{G}_{sp}(s) + D$, where $D = \lim_{s\to\infty} \hat{G}(s)$

$$D = \lim_{s\to\infty} \hat{G}(s) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

- Find the monic least comment denominator of all the entries of $\hat{G}_{sp}(s)$ matrix,

$$d(s) = 1s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n.$$

The monic least common denominator of a family of polynomials is the monic polynomial of the smallest order that can be divided by all those polynomials.

$$\hat{G}_{sp}(s) = \hat{G}(s) - D =$$

$$\begin{bmatrix} \frac{s+1}{s+3} \\ \frac{s-1}{s+1} \\ \frac{s+2}{(s+1)(s+3)} \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{2}{s+3} \\ -\frac{2}{s+1} \\ \frac{s+2}{(s+1)(s+3)} \end{bmatrix}$$

- $\hat{G}_{sp}(s) = \frac{1}{d(s)}[N_1 s^{n-1} + N_2 s^{n-2} + \cdots + N_{n-1}s + N_n],$

$$d(s) = (s+1)(s+3) = 1\,s^2 + \underbrace{4}_{\alpha_1}\,s + \underbrace{3}_{\alpha_2}$$

$$\begin{bmatrix} -\frac{2}{s+3} \\ -\frac{2}{s+1} \\ \frac{s+2}{(s+1)(s+3)} \end{bmatrix} = \frac{1}{s^2+4s+3}\begin{bmatrix} -2(s+1) \\ -2(s+3) \\ s+2 \end{bmatrix} = \frac{1}{s^2+4s+3}\left(\underbrace{\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}}_{N_1} s + \underbrace{\begin{bmatrix} -2 \\ -6 \\ 2 \end{bmatrix}}_{N_2}\right)$$

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From rational proper TF to SS

- $\hat{G}_{sp}(s) = \frac{1}{d(s)}[N_1 s^{n-1} + N_2 s^{n-2} + \cdots + N_{n-1}s + N_n]$,

We claim (controllable canonical form)

$$A = \begin{bmatrix} -\alpha_1 I_{p\times p} & -\alpha_2 I_{p\times p} & \cdots & -\alpha_{n-1} I_{p\times p} & -\alpha_n I_{p\times p} \\ I_{p\times p} & 0_{p\times p} & \cdots & 0_{p\times p} & 0_{p\times p} \\ 0_{p\times p} & I_{p\times p} & \cdots & 0_{p\times p} & 0_{p\times p} \\ \vdots & & \ddots & \vdots & \vdots \\ 0_{p\times p} & 0_{p\times p} & \cdots & I_{p\times p} & 0_{p\times p} \end{bmatrix}_{np\times np}, B = \begin{bmatrix} I_{p\times p} \\ 0_{p\times p} \\ \vdots \\ 0_{p\times p} \\ 0_{p\times p} \end{bmatrix}_{np\times p},$$

$$C = [N_1 \quad N_2 \quad \cdots \quad N_{n-1} \quad N_n]_{q\times np}, \quad D = \lim_{s\to\infty} \hat{G}(s)$$

p: number of inputs

q: number of outputs

$$D = \lim_{s\to\infty} \hat{G}(s) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

We need to show

$$\hat{G}_{sp}(s) = \frac{1}{d(s)}[N_1 s^{n-1} + N_2 s^{n-2} + \cdots + N_{n-1}s + N_n] = C(sI - A)^{-1}B$$

$$d(s) = (s+1)(s+3) = 1\,s^2 + \underbrace{4}_{\alpha_1}\,s + \underbrace{3}_{\alpha_2}$$

$$\begin{bmatrix} -\frac{2}{s+3} \\ -\frac{2}{s+1} \\ \frac{s+2}{(s+1)(s+3)} \end{bmatrix} = \frac{1}{s^2+4s+3}\begin{bmatrix} -2(s+1) \\ -2(s+3) \\ s+2 \end{bmatrix} = \frac{1}{s^2+4s+3}\left(\underbrace{\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}}_{N_1} s + \underbrace{\begin{bmatrix} -2 \\ -6 \\ 2 \end{bmatrix}}_{N_2} \right)$$

$$A = \begin{bmatrix} -4 & -3 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} -2 & -2 \\ -2 & -6 \\ 1 & 2 \end{bmatrix}, D = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

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From rational proper TF to SS

Let $Z = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ \vdots \\ Z_n \end{bmatrix} = (sI - A)^{-1}B$ Then, $\hat{G}_{sp}(s) = C(sI - A)^{-1}B = CZ = N_1 Z_1 + N_2 Z_2 + \cdots + N_n Z_n$

Therefore, we need to show that

$$Z_1 = \frac{s^{n-1}}{d(s)} I_{p\times p},\ Z_2 = \frac{s^{n-2}}{d(s)} I_{p\times p}, \cdots, Z_{n-1} = \frac{s}{d(s)} I_{p\times p},\ Z_n = \frac{1}{d(s)} I_{p\times p}.$$

which can be deduced from the following (recall that $d(s) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n$):

$$Z = \begin{bmatrix} Z_1 \\ Z_2 \\ \cdots \\ Z_n \end{bmatrix} = (sI - A)^{-1}B \Rightarrow (sI - A)Z = B \Rightarrow sZ = AZ + B$$

$$\begin{bmatrix} sZ_1 \\ sZ_2 \\ sZ_3 \\ \vdots \\ sZ_n \end{bmatrix} = \begin{bmatrix} -\alpha_1 I_{p\times p} & -\alpha_2 I_{p\times p} & \cdots & -\alpha_{n-1} I_{p\times p} & -\alpha_n I_{p\times p} \\ I_{p\times p} & 0_{p\times p} & \cdots & 0_{p\times p} & 0_{p\times p} \\ 0_{p\times p} & I_{p\times p} & \cdots & 0_{p\times p} & 0_{p\times p} \\ \vdots & & \ddots & \vdots & \vdots \\ 0_{p\times p} & 0_{p\times p} & \cdots & I_{p\times p} & 0_{p\times p} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ \vdots \\ Z_n \end{bmatrix} + \begin{bmatrix} I_{p\times p} \\ 0_{p\times p} \\ 0_{p\times p} \\ \vdots \\ 0_{p\times p} \end{bmatrix} \Rightarrow$$

$$\begin{cases} sZ_1 = -\alpha_1 Z_1 - \alpha_2 Z_2 - \cdots - \alpha_{n-1} Z_{n-1} - \alpha_n Z_n + I_{p\times p} \\ sZ_2 = Z_1 \Rightarrow Z_2 = \frac{1}{s} Z_1 \\ sZ_3 = Z_2 \Rightarrow Z_3 = \frac{1}{s^2} Z_1 \\ \vdots \\ sZ_n = Z_{n-1} \Rightarrow Z_n = \frac{1}{s^{n-1}} Z_1 \end{cases} \Rightarrow \begin{cases} Z_1 = \frac{s^{n-1}}{d(s)} I_{p\times p} \\ \quad \Downarrow \\ Z_2 = \frac{s^{n-2}}{d(s)} I_{p\times p} \\ \vdots \\ Z_n = \frac{1}{d(s)} I_{p\times p} \end{cases}$$

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From rational proper TF to SS: Example

$$\hat{G}(s)=\begin{bmatrix}\frac{s}{(2s+1)} & \frac{s+1}{(s+2)} & \frac{1}{(s+1)}\\ \frac{s+3}{(s+1)} & \frac{s^2}{(s+\frac{1}{2})(2s+1)} & \frac{-1}{(4s+8)}\end{bmatrix}$$

$p=3$ inputs, $q=2$ outputs

- $D=\lim_{s\to\infty}\begin{bmatrix}\frac{s}{(2s+1)} & \frac{s+1}{(s+2)} & \frac{1}{(s+1)}\\ \frac{s+3}{(s+1)} & \frac{s^2}{(s+\frac{1}{2})(2s+1)} & \frac{-1}{(4s+8)}\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & 1 & 0\\ 1 & \frac{1}{2} & 0\end{bmatrix}$, $\hat{G}_{sp}(s)=\hat{G}(s)-D=\begin{bmatrix}\frac{-1}{4(s+\frac{1}{2})} & \frac{-1}{(s+2)} & \frac{1}{(s+1)}\\ \frac{2}{(s+1)} & \frac{-\frac{1}{2}s-\frac{1}{8}}{(s+\frac{1}{2})(s+\frac{1}{2})} & \frac{-1}{4(s+\frac{1}{2})}\end{bmatrix}$

- $d(s)=\left(s+\frac{1}{2}\right)^2(s+1)(s+2)=s^4+\underbrace{4}_{\alpha_1}s^3+\underbrace{\frac{21}{4}}_{\alpha_2}s^2+\underbrace{\frac{11}{4}}_{\alpha_3}s+\underbrace{½}_{\alpha_4}$

- $\hat{G}_{sp}(s)=\begin{bmatrix}\frac{-1}{4(s+\frac{1}{2})} & \frac{-1}{(s+2)} & \frac{1}{(s+1)}\\ \frac{2}{(s+1)} & \frac{-s-\frac{1}{8}}{(s+\frac{1}{2})(s+\frac{1}{2})} & \frac{-1}{4(s+\frac{1}{2})}\end{bmatrix}=\frac{1}{(s+\frac{1}{2})^2(s+1)(s+2)}\begin{bmatrix}\frac{-1}{4}\left(s+\frac{1}{2}\right)(s+1)(s+2) & -\left(s+\frac{1}{2}\right)^2(s+1) & \left(s+\frac{1}{2}\right)^2(s+2)\\ 2\left(s+\frac{1}{2}\right)^2(s+2) & (-s-\frac{1}{8})(s+1)(s+2) & \frac{-1}{4}\left(s+\frac{1}{2}\right)(s+1)(s+2)\end{bmatrix}=\frac{1}{(s+\frac{1}{2})^2(s+1)(s+2)}$

$$\begin{bmatrix}-s^3/4-(7s^2)/8-(7s)/8-1/4 & -s^3-2s^2-(5s)/4-1/4 & s^3+3s^2+(9s)/4+1/2\\ 2s^3+6s^2+(9s)/2+1 & -s^3/2-(13s^2)/8-(11s)/8-1/4 & -s^3/4-(7s^2)/8-(7s)/8-1/4\end{bmatrix}=$$

$$\frac{1}{(s+\frac{1}{2})^2(s+1)(s+2)}\left(\underbrace{\begin{bmatrix}\frac{-1}{4} & -1 & 1\\ 2 & \frac{-1}{2} & \frac{-1}{4}\end{bmatrix}}_{N_1}s^3+\underbrace{\begin{bmatrix}\frac{-7}{8} & -2 & 3\\ 6 & \frac{-13}{8} & \frac{-7}{8}\end{bmatrix}}_{N_2}s^2+\underbrace{\begin{bmatrix}\frac{-7}{8} & -\frac{5}{4} & \frac{9}{4}\\ \frac{9}{2} & \frac{-11}{8} & \frac{-7}{8}\end{bmatrix}}_{N_3}s+\underbrace{\begin{bmatrix}\frac{-1}{4} & -\frac{1}{4} & \frac{1}{2}\\ 1 & \frac{-1}{4} & \frac{-1}{4}\end{bmatrix}}_{N_4}\right)$$

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From rational proper TF to SS: Example (cont'd)

$$\hat{G}(s)=\begin{bmatrix}\frac{s}{(2s+1)} & \frac{s+1}{(s+2)} & \frac{1}{(s+1)}\\ \frac{s+3}{(s+1)} & \frac{s^2}{(s+\frac{1}{2})(2s+1)} & \frac{-1}{(4s+8)}\end{bmatrix}$$

$$A=\begin{bmatrix}-4 & 0 & 0 & -\frac{21}{4} & 0 & 0 & -\frac{11}{4} & 0 & 0 & -\frac{1}{2} & 0 & 0\\ 0 & -4 & 0 & 0 & -\frac{21}{4} & 0 & 0 & \frac{11}{4} & 0 & 0 & -\frac{1}{2} & 0\\ 0 & 0 & -4 & 0 & 0 & -\frac{21}{4} & 0 & 0 & \frac{11}{4} & 0 & 0 & -\frac{1}{2}\\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\end{bmatrix},\quad B=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$C=\begin{bmatrix}\frac{-1}{4} & -1 & 1 & \frac{-7}{8} & -2 & 3 & \frac{-7}{8} & -\frac{5}{4} & \frac{9}{4} & \frac{-1}{4} & -\frac{1}{4} & \frac{1}{2}\\ 2 & \frac{-1}{2} & \frac{-1}{4} & 6 & \frac{-13}{8} & \frac{-7}{8} & \frac{9}{2} & \frac{-11}{8} & \frac{-7}{8} & 1 & \frac{-1}{4} & \frac{-1}{4}\end{bmatrix},\quad D=\begin{bmatrix}\frac{1}{2} & 1 & 0\\ 1 & \frac{1}{2} & 0\end{bmatrix}$$

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References

[1] Joao P. Hespanha, ``Linear systems theory”, Princeton University Press (Chapter 4)

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