# Linear Systems I
# Lecture 17

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# Review of controllable decomposition

$$
\begin{aligned}
\dot{x} &= Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^p \\
y &= Cx + Du, \quad y \in \mathbb{R}^q
\end{aligned}
$$

## Theorem

$$\text{rank}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = m < n$$

$\exists\, T$ *invertible s.t.* $\bar{x} = T^{-1}x$ *transforms state equations to*

$$
\begin{aligned}
&\bar{A} = T^{-1}AT = \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}, \quad \bar{B} = T^{-1}B = \begin{bmatrix} B_c \\ 0 \end{bmatrix} \\
&\bar{C} = \begin{bmatrix} C_u & C_u \end{bmatrix}, \quad \bar{D} = D, \\
&A_c \in \mathbb{R}^{m \times m}, \quad B_c \in \mathbb{R}^{m \times p}, \quad C_c \in \mathbb{R}^{q \times m},
\end{aligned}
$$

$$T = \Big[ \underbrace{t_1 \quad t_2 \quad \cdots \quad t_m}_{\text{$m$ linearly independent columns of } \mathcal{C}} \Big| \underbrace{t_{m+1} \quad t_{m+2} \quad \cdots \quad t_n}_{\text{any way you can s.t. all columns of $T$ are linearly independent}} \Big]$$

$(A_c, B_c)$ is controllable!

$$\boxed{G(s) = \bar{G}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D} = C_c(sI - A_c)^{-1}B_c + D}$$

# Review of Observable decomposition

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^p$$
$$y = Cx + Du, \quad y \in \mathbb{R}^q$$

**Theorem**

$$\text{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \bar{m} < n :$$

$\exists T$ *invertible s.t.* $\bar{x} = T^{-1}x$ *transforms state equations to*

$$\bar{A} = T^{-1}AT = \begin{bmatrix} A_o & 0 \\ A_{12} & A_{\bar{o}} \end{bmatrix}, \quad \bar{B} = T^{-1}B = \begin{bmatrix} B_o \\ B_{\bar{o}} \end{bmatrix}$$
$$\bar{C} = CT = \begin{bmatrix} C_o & 0 \end{bmatrix}, \quad \bar{D} = D,$$
$$A_o \in \mathbb{R}^{\bar{m} \times \bar{m}}, \quad B_o \in \mathbb{R}^{\bar{m} \times p}, \quad C_o \in \mathbb{R}^{q \times \bar{m}},$$

$$T = \left[ \underbrace{t_1 \quad t_2 \quad \cdots \quad t_{\bar{m}}} \; \middle| \; \underbrace{t_{\bar{m}+1} \quad t_{\bar{m}+2} \quad \cdots \quad t_n} \right]$$

- $t_1, \dots, t_{\bar{m}}$: any way you can s.t. all columns of T are linearly independent
- $t_{\bar{m}+1}, \dots, t_n$: $n - \bar{m}$ linearly independent vectors spanning the nullspace of $\mathcal{O}$

$(A_o, C_o)$ is observable.

$$G(s) = \bar{G}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D} = C_o(sI - A_o)^{-1}B_o + D$$

# Review of Kalman decomposition

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^p, y \in \mathbb{R}^q$$

**Theorem**

$\exists\, T$ *invertible s.t.* $\bar{x} = T^{-1}x$ *transforms state equations to*

$$\text{rank}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = m < n$$

$$\text{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \bar{m} < n$$

$$\begin{bmatrix} \dot{x}_{co} \\ \dot{x}_{c\bar{o}} \\ \dot{x}_{\bar{c}o} \\ \dot{x}_{\bar{c}\bar{o}} \end{bmatrix} = \underbrace{\begin{bmatrix} A_{co} & 0 & A_{\times o} & 0 \\ A_{c\times} & A_{c\bar{o}} & A_{\times\times} & A_{\times\bar{o}} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{\bar{c}\times} & A_{\bar{c}\bar{o}} \end{bmatrix}}_{\bar{A} = T^{-1}AT} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + \underbrace{\begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix}}_{\bar{B} = T^{-1}B} u$$

$$y = \underbrace{\begin{bmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{bmatrix}}_{\bar{C} = CT} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + Du,$$

$$T = \begin{bmatrix} T_{co} & T_{c\bar{o}} & T_{\bar{c}o} & T_{\bar{c}\bar{o}} \end{bmatrix}$$

- columns of $[T_{co} \;\; T_{c\bar{o}}]$ span the $\text{Im}\,\mathcal{C}$
- columns of $T_{c\bar{o}}$ span the $\text{null}\,\mathcal{O} \cap \text{Im}\,\mathcal{C}$
- columns of $[T_{c\bar{o}} \;\; T_{\bar{c}\bar{o}}]$ span the $\text{null}\,\mathcal{O}$
- columns of $T_{\bar{c}o}$ along with the elements described above construct an invertible T
- $(A_{co}, B_{co}, C_{co})$ is both controllable and observable.
- $\left(\begin{bmatrix} A_{co} & 0 \\ A_{cx} & A_{c\bar{o}} \end{bmatrix}, \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \end{bmatrix}\right)$ is controllable
- $\left(\begin{bmatrix} A_{co} & A_{xo} \\ 0 & A_{\bar{c}o} \end{bmatrix}, \begin{bmatrix} C_{co} & C_{\bar{c}o} \end{bmatrix}\right)$ is Observable

$$\boxed{G(s) = \bar{G}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D} = C_{co}(sI - A_{co})^{-1}B_{co} + D}$$

# Review of Lec 4: elementary Realization (from TF rep. to SS rep.)

**Def. (Realization problem)**: how to compute SS representation from a given transfer function.

**Caution**: Note every TF is realizable. Recall that distributed systems have impulse response and as a result transfer function but no SS rep.

**Def. (Realizable TF)**: A transfer function $\hat{G}(s)$ is said to be realizable if there exists a finite dimensional SS equation

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$

or simply $\{A, B, C, D\}$ such that

$$\hat{G}(s) = C(sI - A)^{-1}B + D.$$

We call $\{A, B, C, D\}$ a realization of $\hat{G}(s)$.

**Note**: if a transfer function is realizable it has infinitely many realization, not necessarily of the same dimension.

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**Theorem (realizable transfer function**: A transfer function $\hat{G}(s)$ can be realized by an LTI SS equation iff $\hat{G}(s)$ is a proper rational function.

**Definition (minimum realization)**: A realization of $\hat{G}(s)$ is called minimal or irreducible if there is no realization of $\hat{G}$ of smaller order.

Example:

$$\begin{bmatrix} \frac{4s-10}{2s+1} & \frac{3}{s+2} \\ \frac{1}{(2s+1)(s+2)} & \frac{s+1}{(s+2)^2} \end{bmatrix},$$

The following $(A, B, C, D)$ are all realization of this transfer function:

$$A = \left[\begin{array}{cc|cc|cc} -4.5 & 0 & -6 & 0 & -2 & 0 \\ 0 & -4.5 & 0 & -6 & 0 & -2 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right], \quad B = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right]$$

$$C = \left[\begin{array}{cc|cc|cc} -6 & 3 & -24 & 7.5 & -24 & 3 \\ 0 & 1 & 0.5 & 1.5 & 1 & 0.5 \end{array}\right], \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -2.5 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -4 & -4 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} -6 & -12 & 3 & 6 \\ 0 & 0.5 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -0.4198 & -0.3802 & -0.3654 \\ 0.642 & -3.842 & -3.523 \\ -0.321 & 0.921 & -0.2383 \end{bmatrix}, \quad B = \begin{bmatrix} 0.4 & 0.08889 \\ -0.4 & 0.9111 \\ 0.2 & 0.04444 \end{bmatrix}$$

$$C = \begin{bmatrix} -13.33 & 4.333 & 5.333 \\ 0.5 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

# Minimal Realization of a TF

**Definition (minimum realization)**: A realization of $\hat{G}(s)$ is called minimal or irreducible if there is no realization of $\hat{G}$ of smaller order.

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**Theorem** Every minimal realization must be both controllable and observable.

Hint for proof: remember Kalman decomposition

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**Theorem**

*A realization is minimal if and only if it is both controllable and observable.*

We invoke the following results in the proof

- **Theorem** Two realizations

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \qquad \begin{cases} \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \\ y = \bar{C}\bar{x} + \bar{D}u \end{cases}$$

  are zero-state equivalent if and only if

$$D = \bar{D}, \quad CA^i B = \bar{C}\bar{A}^i\bar{B}, \forall i \geqslant 0$$

- For two matrices $M \in R^{r\times q}$ and $N \in Rq \times p$: $\text{rank}(MN) \leqslant \min\{\text{rank}(M), \text{rank}(N)\}$
- **Theorem** Algebraically equivalent systems have same transfer function.

**Theorem** All minimal realizations of a transfer function are algebraically equivalent.

# Order of a minimal SISO realization

## Theorem

***Theorem*** *Consider* $\hat{g}(s) = \frac{n(s)}{d(s)}$*,where* $d(s)$ *is a monic polynomial and* $n(s)$ *and* $d(s)$ *are coprime.*
*A SISO realization*

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \quad x \in R^n, \; u, y \in R$$

*of* $\hat{g}(s)$ *is minimal if and only if its order* $n$ *is equal to the degree of* $\hat{g}(s)$*.*

- *In this case, the pole polynomial* $d(s)$ *of* $\hat{g}(s)$ *is equal to the characteristic polynomial of* $A$*; i.e.,* $d(s) = det(sI - A)$*.*

## Theorem

*Assuming that the SISO realization of* $\hat{g}(s)$ *is minimal, the transfer function* $\hat{g}(s)$ *is BIBO stable if and only if the realization is (internally) asymptotically stable.*

# Order of a minimal SISO realization (proof of the main theorem)

**Theorem: A SISO realization** $\dot{x} = Ax + Bu, \quad y = Cx + Du, \quad x \in \mathbb{R}^n, \; u, y \in \mathbb{R}$

**of** $\hat{g}(s) = \frac{n(s)}{d(s)}$**, where** $n(s)$ **and** $d(s)$ **are coprime, is minimal if and only if its order** $n$ **is equal to the degree of** $\hat{g}(s)$**. In this case, the pole polynomial** $d(s)$ **of** $\hat{g}(s)$ **is equal to the characteristic polynomial of** $A$**; i.e.,** $d(s) = \det(sI - A)$.

Since, the direct gain D of a realization does not affect its minimal realization, we can ignore it in the proof. We assume that $\hat{g}(s)$ is strictly proper and can be represented as

$$\hat{g}(s) = \frac{n(s)}{d(s)} = \frac{\beta_1 s^{n-1} + \beta_2 s^{n-2} + \cdots + \beta_{n-1} s + \beta_n}{s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_{n-1} s + \alpha_n},$$

The proof needs only to show that $\hat{g}(s)$ has a realization of order $n$ that is both controllable and observable (recall that a realization is minimal if and only if it is both controllable and observable). In earlier lectures we showed that the following is a realization of $\hat{g}(s)$. This realization is called controllable canonical form.

$$A = \begin{bmatrix} -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} & -\alpha_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad C = [\beta_1 \quad \beta_2 \quad \cdots \quad \beta_{n-1} \quad \beta_n], \quad D = d;$$

you have already shown in one of your HWs that $(A, B)$ is controllable. We only need to show that $(A, C)$ is observable too. For this let use PBH eigenvector test for observability. Let $x = [x_1, x_2, \cdots, x_n]^\top \neq 0$ be an eigenvector of $A$, i.e.,

$$Ax = \lambda x \quad \Leftrightarrow \begin{cases} -\sum_{i=1}^n \alpha_i x_i = \lambda x_1, \\ x_1 = \lambda x_2 \\ x_2 = \lambda x_3 \\ \vdots \\ x_{n-1} = \lambda x_n \end{cases} \Leftrightarrow \begin{cases} -\sum_{i=1}^n \alpha_i \lambda^{n-i} x_n = \lambda^n x_n, \\ x_1 = \lambda^{n-1} x_n \\ x_2 = \lambda^{n-2} x_n \\ \vdots \\ x_{n-1} = \lambda x_n \end{cases} \Leftrightarrow \begin{cases} d(\lambda) x_n = 0 \quad (\star), \\ x_1 = \lambda^{n-1} x_n \\ x_2 = \lambda^{n-2} x_n \\ \vdots \\ x_{n-1} = \lambda x_n \end{cases}$$

Because $x = [x_1, x_2, \cdots, x_{n-1}, x_n]^\top = [\lambda^{n-1} x_n, \lambda^{n-2} x_2, \cdots, \lambda x_n, x_n]^\top \neq 0$ then $x_n$ has to be different that zero. Then , from $(\star)$ we have that $d(\lambda) = 0$, i.e, $\lambda$ is a root of $d(s)$. On the other hand,

$$Cx = \sum_{i=1}^n \beta_i x_i = \sum_{i=1}^n \beta_i \lambda^{n-i} x_n = n(\lambda) x_n$$

Since $d(s)$ and $n(s)$ are coprime and $\lambda$ is a root of $d(s)$, it cannot be a root of $n(s)$, i.e., $n(\lambda) \neq 0$. Since $x_n \neq 0$, then $Cx \neq 0$, and therefore, $(A, C)$ must be observable.

# Order of a minimal SISO realization: numerical example

$$A = \begin{bmatrix} -1 & 0 \\ a & 2 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \; C = \begin{bmatrix} 2 & 3 \end{bmatrix}, \; D = 0.$$

Q: For what values of $a$ this system is a minimal realization?

$$\text{rank}\begin{bmatrix} B & AB \end{bmatrix} = \text{rank}\begin{bmatrix} 1 & -1 \\ -1 & a-2 \end{bmatrix} = 2, \text{ unless } a = 3,$$

$$\text{rank}\begin{bmatrix} C \\ CA \end{bmatrix} = \text{rank}\begin{bmatrix} 2 & 3 \\ -2+3a & 6 \end{bmatrix} = 2, \text{ unless } a = 2,$$

If $a = 2$ or $a = 3$, $(A, B, C, D)$ is not a minimal representation

$$\hat{g}(s) = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} s+1 & 0 \\ -a & s-2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{a}{(s+1)(s-2)} & \frac{1}{s-2} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} =$$

$$\frac{-(s+7-3a)}{(s+1)(s-2)} = \begin{cases} \frac{-(s+1)}{(s+1)(s-2)} = \frac{-1}{(s-2)} & a = 2 \\ \frac{-(s-2)}{(s+1)(s-2)} = \frac{-1}{(s+1)} & a = 3 \end{cases}$$

# State estimation (asymptotic observer)

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$

**The simplest state estimator is:** $\dot{\hat{x}} = A\hat{x} + Bu$

**We want** $\lim_{t\to\infty} \hat{x}(t) \to x(t)$. **To study the performance, let us look at error dynamics and its evolution in time**

$$e(t) := \hat{x}(t) - x(t) \Rightarrow \dot{e} = A\hat{x} + Bu - Ax - Bu = Ae \Rightarrow \dot{e} = Ae.$$

**If** $A$ **is a stability matrix (all its eigenvalues have strictly negative real part), we have** $\lim_{t\to\infty} e(t) \to 0$, **for every input.**

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**When** $A$ **is not a stability matrix, it is still possible to construct an asymptotic correct state estimator by modifying the observer dynamics as follows**

$$\dot{\hat{x}} = A\hat{x} + Bu - L(\hat{y} - y), \quad \hat{y} = C\hat{x} + Du, \quad (L : \textbf{output injection matrix gain}).$$

**In this case error dynamics is given by**

$$e(t) := \hat{x}(t) - x(t) \Rightarrow \dot{e} = A\hat{x} + Bu - L(C\hat{x} + Du - Cx - Du) - Ax - Bu \Rightarrow \dot{e} = (A - LC)e$$

Theorem: **If the output injection matrix** $L$ **makes** $A - LC$ **a stability matrix, then** $\lim_{t\to\infty} e(t) \to 0$ **exponentially fast, for every input** $u$.

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Theorem: **When** $(A, C)$ **is observable, it is always possible to find a matrix** $L$ **such that** $A - LC$ **is a stability matrix. (we will show later that this also possible when** $(A, C)$ **is detectible.)**

Theorem: **When** $(A, C)$ **is observable, given any** $n$ **symmetric set of complex numbers** $\{\nu_1, \nu_2, \cdots, \nu_n\}$, **there exists a** $L$ **such that** $A - LC$ **has eigenvalues equal to** $\{\nu_1, \nu_2, \cdots, \nu_n\}$.

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## Procedure to design output injection matrix gain

$$\begin{cases} (A, C) \textbf{ observable } \Leftrightarrow (A^\top, C^\top) \textbf{ observable}, \\ \textbf{eig}(A - LC) = \textbf{eig}(A - LC)^\top = \textbf{eig}(A^\top - C^\top L^\top), \end{cases} \Rightarrow$$

$$\begin{cases} \textbf{Let } \bar{A} = A^\top,\ \bar{B} = C^\top,\ \bar{K} = L^\top :\ \textbf{eig}(A - LC) = \textbf{eig}(\bar{A} - \bar{B}\bar{K}), \\ \textbf{use tools from state-feedback design to obtain } \bar{K} \textbf{ that stabilizes } (\bar{A} - \bar{B}\bar{K}), \end{cases} \Rightarrow L = \bar{K}^\top \textbf{ stabilizes } (A - LC)$$

## Next Lecture

# Stabilization through output feedback

**system**: $\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases}$

**observer**: $\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - L(\hat{y}(t) - y(t)), \\ \hat{y}(t) = C\hat{x}(t) + Du(t), \end{cases}$ ,

**control**: $u = -K\hat{x}$

**To study whether the closed loop of the systems above is stable, we construct a state-space model for the closed-loop system using states** $\bar{x} := \begin{bmatrix} x \\ e \end{bmatrix}$, **where** $e = \hat{x} - x$. **We obtain**

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \underbrace{\begin{bmatrix} A - BK & -BK \\ 0 & A - LC \end{bmatrix}}_{\bar{A}} \begin{bmatrix} x \\ e \end{bmatrix}.$$

**We want** $\bar{A}$ **to be a stability matrix, i.e., all eigenvalues of** $\bar{A}$ **have strictly negative real parts**

$$\textbf{eig}(\bar{A}) = \{\textbf{eig}(A - BK), \textbf{eig}(A - LC)\}$$

**Procedure to design $L$ and $K$ for stabilization through output feedback**: **design** $L$ **that stabilizes** $A - LC$, **and design** $K$ **that stabilizes** $A - BK$. **These two designs are independent from one another (separation in design) and are possible if** $(A, B, C)$ **is controllable and observable (indeed the tasks can be achieved if** $(A, B, C)$ **is stablilizable and detectible).**

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**Notice** : $\begin{bmatrix} x \\ e \end{bmatrix} = \underbrace{\begin{bmatrix} I_n & 0 \\ -I_n & I_n \end{bmatrix}}_{T} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$. **Because** $T$ **is invertible, it is a similarity transformation matrix. That is, LTI system with states** $(x, e)$ **is algebraically equivalent to LTI system with states** $(x, \hat{x})$.