Linear Systems I
Lecture 16

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Observability of LTI systems

$$\begin{cases} \dot{x} = Ax + Bu, & x \in \mathbb{R}^n, \; u \in \mathbb{R}^p \\ y = Cx + Du, & y \in \mathbb{R}^q \end{cases} \qquad x(0) = x_0 \in \mathbb{R}^n \qquad (\star)$$

Question of interest in Observability: Can we reconstruct $x(0)$ by knowing $y(t)$ and $u(t)$ over some finite time interval $[0, t_1]$? (By knowing the initial condition, we can reconstruct the entire state $x(t)$, then use it in our state feedback to control the system)

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \Leftrightarrow \bar{y}(t) = Ce^{At}x(0)$$

$$\bar{y}(t) = y(t) - C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau - Du(t)$$

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The LTI state-space equation $(\star)$ is said **to be observable** if for any unknown initial state $x(0)$, $\exists$ finite time $t_1 > 0$ such that the knowledge of the input u and the output y over $[0, t_1]$ suffices to determine uniquely the initial state $x(0)$. Otherwise, the equation is said to be unobservable.

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Observability gramian

$$\begin{cases} \dot{x} = Ax + Bu, & x \in \mathbb{R}^n,\ u \in \mathbb{R}^p \\ y = Cx + Du, & y \in \mathbb{R}^q \end{cases} \qquad x(0) = x_0 \in \mathbb{R}^n \qquad (\star)$$

$$\underbrace{\bar{y}(t)}_{\mathbb{R}^q} = \underbrace{Ce^{At}}_{\mathbb{R}^{q\times n}} \underbrace{x(0)}_{\mathbb{R}^n}$$

$$\underbrace{(Ce^{At})^\top \bar{y}(t)}_{\mathbb{R}^n} = \underbrace{(Ce^{At})^\top\, Ce^{At}}_{\mathbb{R}^{n\times n}} \underbrace{x(0)}_{\mathbb{R}^n}$$

Using input output information over $[0, t]$ we obtain

$$\underbrace{\int_0^t e^{A^\top \tau} C^\top \bar{y}(\tau) d\tau}_{\text{known}} = \underbrace{W_O(t)}_{\text{known}} \underbrace{x(0)}_{\text{unknown}},$$

Observability gramian: $W_O(t) = \int_0^t e^{A^\top \tau} C^\top\, Ce^{A\tau} d\tau$

- $\text{rank}(W_O(t)) = n \Rightarrow$ unique x_0 can be obtained: system is observable
- $\text{rank}(W_O(t)) < n \Rightarrow x_0$ is not unique: system is not observable
  - (if $x_0 \in \text{Ker}(W_O(t))$, then $W_O(t)x_0 = 0$): unobservable subspace $\text{Ker}(W_O(t))$.

Duality Theorem

$$\begin{cases} \dot{x} = Ax + Bu, & x \in \mathbb{R}^n,\ u \in \mathbb{R}^p \\ y = Cx + Du, & y \in \mathbb{R}^q \end{cases} \qquad x(0) = x_0 \in \mathbb{R}^n \qquad (\star)$$

Theorem

The pair (A, C) is observable if and only if the pair $(A^\top, C^\top)$ is controllable.

- (A, C) is observable iff
  $W_O(t) = \int_0^t e^{A^\top \tau} C^\top C e^{A\tau} d\tau$ is full rank
- $(A^\top, C^\top)$ is controllable iff
  $W_C(t) = \int_0^t e^{A^\top \tau} C^\top (C^\top)^\top e^{(A^\top)^\top \tau} d\tau = \int_0^t e^{A^\top \tau} C^\top C e^{A\tau} d\tau$ is full rank

Note that

$$W_O(t) = W_C(t)$$

Tests for Observability of LTI systems

The following statements are equivalent:

1. the n-dimentional pair (A, C) is observable
2. The $n \times n$ matrix $\boxed{W_O(t) = \int_0^t e^{A^\top \tau} C^\top C e^{A\tau} d\tau}$ is nonsingular for all $t > 0$.
3. Let $\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}_{nq \times n}$ be the observability matrix, then $\text{rank}(\mathcal{O}) = n$
4. $\text{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$ for all complex λ
5. $\text{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$ for all λ eigenvalues of A
6. If in addition, all eigenvalues of A have negative real parts, then the unique solution of
$$A^\top W_O + W_O A = -C^\top C$$
is positive definite. The solution is called the observability Gramian and can be expressed as
$$W_O = \int_0^\infty e^{A^\top \tau} C^\top C e^{A\tau} d\tau$$

Recall that for any matrix L, $\text{rank}(L) = \text{rank}(L^\top)$

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^p$$
$$y = Cx + Du, \quad y \in \mathbb{R}^q$$

Theorem

$$\text{rank}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = m < n$$

$\exists\, T$ *invertible s.t.* $\bar{x} = T^{-1}x$ *transforms state equations to*

$$\bar{A} = T^{-1}AT = \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}, \quad \bar{B} = T^{-1}B = \begin{bmatrix} B_c \\ 0 \end{bmatrix}$$
$$\bar{C} = \begin{bmatrix} C_u & C_u \end{bmatrix}, \quad \bar{D} = D,$$
$$A_c \in \mathbb{R}^{m\times m}, \quad B_c \in \mathbb{R}^{m\times p}, \quad C_c \in \mathbb{R}^{q\times m},$$

$$T = [\ \underbrace{t_1 \quad t_2 \quad \cdots \quad t_m}\ \big|\ \underbrace{t_{m+1} \quad t_{m+2} \quad \cdots \quad t_n}\ ]$$

$t_1, \ldots, t_m$: m linearly independent columns of $\mathcal{C}$

$t_{m+1}, \ldots, t_n$: any way you can s.t. all columns of T are linearly independent

(A_c, B_c) is controllable!

$$G(s) = \bar{G}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D} = C_c(sI - A_c)^{-1}B_c + D$$

Observable decomposition

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^p$$
$$y = Cx + Du, \quad y \in \mathbb{R}^q$$

Theorem

$$\text{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \bar{m} < n :$$

$\exists T$ *invertible s.t.* $\bar{x} = T^{-1}x$ *transforms state equations to*

$$\bar{A} = T^{-1}AT = \begin{bmatrix} A_o & 0 \\ A_{12} & A_{\bar{o}} \end{bmatrix}, \quad \bar{B} = T^{-1}B = \begin{bmatrix} B_o \\ B_{\bar{o}} \end{bmatrix}$$

$$\bar{C} = CT = \begin{bmatrix} C_o & 0 \end{bmatrix}, \quad \bar{D} = D,$$

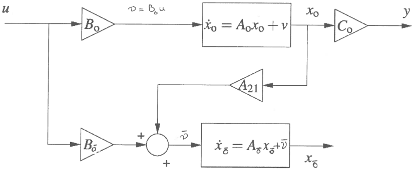
$$A_o \in \mathbb{R}^{\bar{m}\times\bar{m}}, \quad B_o \in \mathbb{R}^{\bar{m}\times p}, \quad C_o \in \mathbb{R}^{q\times\bar{m}},$$

$$T = \Big[\underbrace{t_1 \quad t_2 \quad \cdots \quad t_{\bar{m}}}_{} \;\Big|\; \underbrace{t_{\bar{m}+1} \quad t_{\bar{m}+2} \quad \cdots \quad t_n}_{} \Big]$$

- $t_1, \dots, t_{\bar{m}}$: any way you can s.t. all columns of T are linearly independent
- $t_{\bar{m}+1}, \dots, t_n$: $n - \bar{m}$ linearly independent vectors spanning the nullspace of $\mathcal{O}$

(A_o, C_o) is observable.

$$G(s) = \bar{G}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D}$$
$$= C_o(sI - A_o)^{-1}B_o + D$$



Detectability

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^p$$
$$y = Cx + Du, \quad y \in \mathbb{R}^q$$

Theorem

$$\text{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \bar{m} < n :$$

$\exists T$ *invertible s.t.* $\bar{x} = T^{-1}x$ *transforms state equations to*

$$\dot{\bar{x}} = \begin{bmatrix} \dot{x}_o \\ \dot{x}_{\bar{o}} \end{bmatrix} = \underbrace{\begin{bmatrix} A_o & 0 \\ A_{12} & A_{\bar{o}} \end{bmatrix}}_{\bar{A} = T^{-1}AT} \begin{bmatrix} x_o \\ x_{\bar{o}} \end{bmatrix} + \underbrace{\begin{bmatrix} B_o \\ B_{\bar{o}} \end{bmatrix}}_{\bar{B} = T^{-1}B} u$$

$$y = \underbrace{\begin{bmatrix} C_o & 0 \end{bmatrix}}_{\bar{C} = CT} \begin{bmatrix} x_o \\ x_{\bar{o}} \end{bmatrix} + Du,$$

$$A_o \in \mathbb{R}^{\bar{m} \times \bar{m}}, \quad B_o \in \mathbb{R}^{\bar{m} \times p}, \quad C_o \in \mathbb{R}^{q \times \bar{m}},$$

$$\dot{x}_{\bar{o}} = A_{\bar{o}} x_{\bar{o}} + A_{21} x_o + B_{\bar{o}} u \Rightarrow x_{\bar{o}}(t) = e^{A_{\bar{o}}(t-t_0)} x_{\bar{o}}(0) + \int_{t_0}^{t} e^{A_{\bar{o}}(t-\tau)} (A_{21} x_o(\tau) + B_{\bar{o}} u(\tau)) d\tau$$

- (A_o, C_o) is observable, i.e., x_o can be reconstructed from input and output, then
- if $A_{\bar{o}}$ is a stability matrix, $\lim_{t \to \infty} e^{A_{\bar{o}}(t-t_0)} x_{\bar{o}}(0) \to 0$: $x_{\bar{o}}$ can be guessed to an error that converges to zero exponentially fast.

Def. The pair (A, C) is detectible if it is algebraically equivalent to a system in the standard form for unobservable systems with $n = \bar{m}$ (i.e., $A_{\bar{o}}$ nonexistent) or with $A_{\bar{o}}$ a stability matrix.

Next lecture(s)

Kalman decomposition

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^p, y \in \mathbb{R}^q$$

Theorem

$$\mathrm{rank}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = m < n$$

$$\mathrm{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \bar{m} < n$$

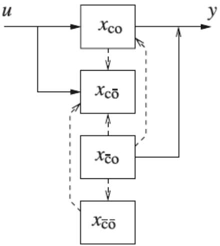
$\exists T$ *invertible s.t.* $\bar{x} = T^{-1}x$ *transforms state equations to*

$$\begin{bmatrix} \dot{x}_{co} \\ \dot{x}_{c\bar{o}} \\ \dot{x}_{\bar{c}o} \\ \dot{x}_{\bar{c}\bar{o}} \end{bmatrix} = \underbrace{\begin{bmatrix} A_{co} & 0 & A_{\times o} & 0 \\ A_{c\times} & A_{c\bar{o}} & A_{\times\times} & A_{\times\bar{o}} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{\bar{c}\times} & A_{\bar{c}\bar{o}} \end{bmatrix}}_{\bar{A} = T^{-1}AT} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + \underbrace{\begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix}}_{\bar{B} = T^{-1}B} u$$

$$y = \underbrace{\begin{bmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{bmatrix}}_{\bar{C} = CT} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + Du,$$

$$T = \begin{bmatrix} T_{co} & T_{c\bar{o}} & T_{\bar{c}o} & T_{\bar{c}\bar{o}} \end{bmatrix}$$

- columns of $[T_{co} \;\; T_{c\bar{o}}]$ span the $\mathrm{Im}\,\mathcal{C}$
- columns of $T_{c\bar{o}}$ span the $\mathrm{null}\,\mathcal{O} \cap \mathrm{Im}\,\mathcal{C}$
- columns of $[T_{c\bar{o}} \;\; T_{\bar{c}\bar{o}}]$ span the $\mathrm{null}\,\mathcal{O}$
- columns of $T_{\bar{c}o}$ along with the elements described above construct an invertible T



- (A_{co}, B_{co}, C_{co}) is both controllable and observable.
- $\left(\begin{bmatrix} A_{co} & 0 \\ A_{c\times} & A_{c\bar{o}} \end{bmatrix}, \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \end{bmatrix}\right)$ is controllable
- $\left(\begin{bmatrix} A_{co} & A_{\times o} \\ 0 & A_{\bar{c}o} \end{bmatrix}, \begin{bmatrix} C_{co} & C_{\bar{c}o} \end{bmatrix}\right)$ is controllable

$$G(s) = \bar{G}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D} = C_{co}(sI - A_{co})^{-1}B_{co} + D$$

Review of Lec 4: elementary Realization (from TF rep. to SS rep.)

Def. (Realization problem): how to compute SS representation from a given transfer function.

Caution: Note every TF is realizable. Recall that distributed systems have impulse response and as a result transfer function but no SS rep.

Def. (Realizable TF): A transfer function $\hat{G}(s)$ is said to be realizable if there exists a finite dimensional SS equation

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$

or simply $\{A, B, C, D\}$ such that

$$\hat{G}(s) = C(sI - A)^{-1}B + D.$$

We call $\{A, B, C, D\}$ a realization of $\hat{G}(s)$.

Note: if a transfer function is realizable it has infinitely many realization, not necessarily of the same dimension.

Theorem (realizable transfer function): A transfer function $\hat{G}(s)$ can be realized by an LTI SS equation iff $\hat{G}(s)$ is a proper rational function.

Minimal Realization of a TF

Definition (minimum realization): A realization of $\hat{G}(s)$ is called minimal or irreducible if there is no realization of $\hat{G}$ of smaller order.

Theorem A realization is minimal if and only if it is both controllable and observable.

Theorem All minimal realizations of a transfer function are algebraically equivalent.

Order of a minimal SISO realization

Theorem: A SISO realization $\dot{x} = Ax + Bu, \quad y = Cx + Du, \quad x \in \mathbb{R}^n, \; u, y \in \mathbb{R}$

of $\hat{g}(s) = \frac{n(s)}{d(s)}$**, where** $n(s)$ **and** $d(s)$ **are coprime, is minimal if and only if its order** n **is equal to the degree of** $\hat{g}(s)$**. In this case, the pole polynomial** $d(s)$ **of** $\hat{g}(s)$ **is equal to the characteristic polynomial of** A**; i.e.,** $d(s) = \det(sI - A)$**.**

Since, the direct gain D of a realization does not affect its minimal realization, we can ignore it in the proof. We assume that $\hat{g}(s)$ is strictly proper and can be represented as

$$\hat{g}(s) = \frac{n(s)}{d(s)} = \frac{\beta_1 s^{n-1} + \beta_2 s^{n-2} + \cdots + \beta_{n-1} s + \beta_n}{s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_{n-1} s + \alpha_n},$$

The proof needs only to show that $\hat{g}(s)$ has a realization of order n that is both controllable and observable (recall that a realization is minimal if and only if it is both controllable and observable). In earlier lectures we showed that the following is a realization of $\hat{g}(s)$. This realization is called controllable canonical form.

$$A = \begin{bmatrix} -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} & -\alpha_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \end{bmatrix}, \quad D = d;$$

you have already shown in one of your HWs that (A, B) is controllable. We only need to show that (A, C) is observable too. For this let use PBH eigenvector test for observability. Let $x = [x_1, x_2, \cdots, x_n]^\top \neq 0$ be an eigenvector of A, i.e.,

$$Ax = \lambda x \quad \Leftrightarrow \quad \begin{cases} -\sum_{i=1}^n \alpha_i x_i = \lambda x_1, \\ x_1 = \lambda x_2 \\ x_2 = \lambda x_3 \\ \vdots \\ x_{n-1} = \lambda x_n \end{cases} \Leftrightarrow \begin{cases} -\sum_{i=1}^n \alpha_i \lambda^{n-i} x_n = \lambda^n x_n, \\ x_1 = \lambda^{n-1} x_n \\ x_2 = \lambda^{n-2} x_n \\ \vdots \\ x_{n-1} = \lambda x_n \end{cases} \Leftrightarrow \begin{cases} d(\lambda)\, x_n = 0 \quad (\star), \\ x_1 = \lambda^{n-1} x_n \\ x_2 = \lambda^{n-2} x_n \\ \vdots \\ x_{n-1} = \lambda x_n \end{cases}$$

Because $x = [x_1, x_2, \cdots, x_{n-1}, x_n]^\top = [\lambda^{n-1} x_n, \lambda^{n-2} x_2, \cdots, \lambda x_n, x_n]^\top \neq 0$ then x_n has to be different that zero. Then , from $(\star)$ we have that $d(\lambda) = 0$, i.e, λ is a root of $d(s)$. On the other hand,

$$Cx = \sum_{i=1}^n \beta_i x_i = \sum_{i=1}^n \beta_i \lambda^{n-i} x_n = n(\lambda) x_n$$

Since $d(s)$ and $n(s)$ are coprime and λ is a root of $d(s)$, it cannot be a root of $n(s)$, i.e., $n(\lambda) \neq 0$. Since $x_n \neq 0$, then $Cx \neq 0$, and therefore, (A, C) must be observable.

Order of a minimal SISO realization: numerical example

$$A = \begin{bmatrix} -1 & 0 \\ a & 2 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \; C = \begin{bmatrix} 2 & 3 \end{bmatrix}, \; D = 0.$$

$$\text{rank}\begin{bmatrix} B & AB \end{bmatrix} = \text{rank}\begin{bmatrix} 1 & -1 \\ -1 & a-2 \end{bmatrix} = 2, \text{ unless } a = 3,$$

$$\text{rank}\begin{bmatrix} C \\ CA \end{bmatrix} = \text{rank}\begin{bmatrix} 2 & 3 \\ -2+3a & 6 \end{bmatrix} = 2, \text{ unless } a = 2,$$

If $a = 2$ or $a = 3$, (A, B, C, D) is not a minimal representation

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$$\hat{g}(s) = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} s+1 & 0 \\ -a & s-2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{a}{(s+1)(s-2)} & \frac{1}{s-2} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} =$$

$$\frac{-(s+7-3a)}{(s+1)(s-2)} = \begin{cases} \frac{-(s+1)}{(s+1)(s-2)} = \frac{-1}{(s-2)}, & \text{if } a = 2 \\ \frac{-(s-2)}{(s+1)(s-2)} = \frac{-1}{(s+1)}, & \text{if } a = 3 \end{cases}$$

Notice that for $a = 2$ or $a = 3$ the degree of the transfer function is 1, and is not equal to the order of A matrix, which is 2. Therefore, for $a = 2$ and $a = 3$ the given realization above is not minimal.

State estimation (asymptotic observer)

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$

The simplest state estimator is: $\dot{\hat{x}} = A\hat{x} + Bu$

We want $\lim_{t\to\infty} \hat{x}(t) \to x(t)$. **To study the performance, let us look at error dynamics and its evolution in time**

$$e(t) := \hat{x}(t) - x(t) \Rightarrow \dot{e} = A\hat{x} + Bu - Ax - Bu = Ae \Rightarrow \dot{e} = Ae.$$

If A **is a stability matrix (all its eigenvalues have strictly negative real part), we have** $\lim_{t\to\infty} e(t) \to 0$, **for every input.**

When A **is not a stability matrix, it is still possible to construct an asymptotic correct state estimator by modifying the observer dynamics as follows**

$$\dot{\hat{x}} = A\hat{x} + Bu - L(\hat{y} - y), \quad \hat{y} = C\hat{x} + Du, \quad (L : \textbf{output injection matrix gain}).$$

In this case error dynamics is given by

$$e(t) := \hat{x}(t) - x(t) \Rightarrow \dot{e} = A\hat{x} + Bu - L(C\hat{x} + Du - Cx - Du) - Ax - Bu \Rightarrow \dot{e} = (A - LC)e$$

Theorem: **If the output injection matrix** L **makes** $A - LC$ **a stability matrix, then** $\lim_{t\to\infty} e(t) \to 0$ **exponentially fast, for every input** u.

Theorem: **When** (A, C) **is observable, it is always possible to find a matrix** L **such that** $A - LC$ **is a stability matrix. (we will show later that this also possible when** (A, C) **is detectible.)**

Theorem: **When** (A, C) **is observable, given any** n **symmetric set of complex numbers** $\{v_1, v_2, \cdots, v_n\}$, **there exists a** L **such that** $A - LC$ **has eigenvalues equal to** $\{v_1, v_2, \cdots, v_n\}$.

Procedure to design output injection matrix gain

$$\begin{cases} (A, C) \textbf{ observable } \Leftrightarrow (A^\top, C^\top) \textbf{ observable}, \\ \textbf{eig}(A - LC) = \textbf{eig}(A - LC)^\top = \textbf{eig}(A^\top - C^\top L^\top), \end{cases} \Rightarrow$$

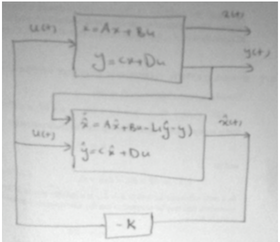
$$\begin{cases} \textbf{Let } \bar{A} = A^\top,\ \bar{B} = C^\top,\ \bar{K} = L^\top : \ \textbf{eig}(A - LC) = \textbf{eig}(\bar{A} - \bar{B}\bar{K}), \\ \textbf{use tools from state-feedback design to obtain } \bar{K} \textbf{ that stabilizes } (\bar{A} - \bar{B}\bar{K}), \end{cases} \Rightarrow L = \bar{K}^\top \textbf{ stabilizes } (A - LC)$$

Stabilization through output feedback

system: $\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases}$

observer: $\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - L(\hat{y}(t) - y(t)), \\ \hat{y}(t) = C\hat{x}(t) + Du(t), \end{cases}$,

control: $u = -K\hat{x}$



To study whether the closed loop of the systems above is stable, we construct a state-space model for the closed-loop system using states $\bar{x} := \begin{bmatrix} x \\ e \end{bmatrix}$**, where** $e = \hat{x} - x$**. We obtain**

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \underbrace{\begin{bmatrix} A - BK & -BK \\ 0 & A - LC \end{bmatrix}}_{\bar{A}} \begin{bmatrix} x \\ e \end{bmatrix}.$$

We want $\bar{A}$ **to be a stability matrix, i.e., all eigenvalues of** $\bar{A}$ **have strictly negative real parts**

$$\mathbf{eig}(\bar{A}) = \{\mathbf{eig}(A - BK), \mathbf{eig}(A - LC)\}$$

Procedure to design L and K for stabilization through output feedback: design L that stabilizes $A - LC$, and design K that stabilizes $A - BK$. These two designs are independent from one another (separation in design) and are possible if (A, B, C) is controllable and observable (indeed the tasks can be achieved if (A, B, C) is stablilizable and detectible).

Notice : $\begin{bmatrix} x \\ e \end{bmatrix} = \underbrace{\begin{bmatrix} I_n & 0 \\ -I_n & I_n \end{bmatrix}}_{T} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$. **Because T is invertible, it is a similarity transformation matrix. That is, LTI system with states (x, e) is algebraically equivalent to LTI system with states $(x, \hat{x})$.**