Linear Systems I
Lecture 14

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Regulation via state-feedback control

Consider $\dot{x} = Ax + Bu, \quad x(0) = x_0 \neq 0 \in \mathbb{R}^n$

Definition (Regulation problem)

Starting from nonzero initial conditions, force the state vector to zero as $t \to \infty$.

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Goal: We want to solve this problem using state feedback $u = -Kx$

$$\begin{aligned} &\dot{x} = A_{cl}x, \quad A_{cl} = (A - BK) \in \mathbb{R}^{n \times n}, K \in \mathbb{R}^{n \times p}, \\ &x(0) = x_0 \neq 0 \in \mathbb{R}^n. \end{aligned}$$

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$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

- A is Hurwitz, regulation can be solved using $u = 0$
- We want some performance
    - how fast
    - certain transient response
    - minimum energy,
    - etc

Review: any controllable LTI system can be stabilized by full state feedback

Consider $\dot{x} = Ax + Bu, \quad x(0) = x_0 \neq 0 \in \mathbb{R}^n \quad (\star)$

Objective If (A, B) is controllable, then we can stabilize system $(\star)$ using full state feedback control $u = -Kx$, i.e., in $\dot{x} = Ax + Bu = (A - BK)x = A_cl\, x$, the closed-loop matrix A_{cl} is Hurwitz.

- For any $\mu \in \mathbb{R}$, if (A, B) controllable, then $(-A - \mu I, B)$ is also controllable.
- $\sigma(-A - \mu I) = \{\lambda_i - \mu\}_{i=1}^N$ where $\{\lambda_i\}_{i=1}^N = \sigma(A)$
- we can always find $\mu \in \mathbb{R}_{>0}$ such that $-A - \mu I$ is Hurwitz.
- there exists a $W > 0$[1] such that $(-A - \mu I)W + W(-A - \mu I)^\top = -BB^\top$
- let $P = W^{-1}$ then you can write

$$PA + A^\top P - PBB^\top P = -2\mu P$$

- let $K = \frac{1}{2}B^\top P$, then we can write

$$P(A - BK) + (A - BK)^\top P = -2\mu P < 0 \Rightarrow \text{ A − BK is a stability matrix}$$

[1] $W = \int_0^\infty \mathbf{e}^{(-\mu I - A)\tau} B B^\top \mathbf{e}^{(-\mu I - A)^\top \tau} d\tau > 0$

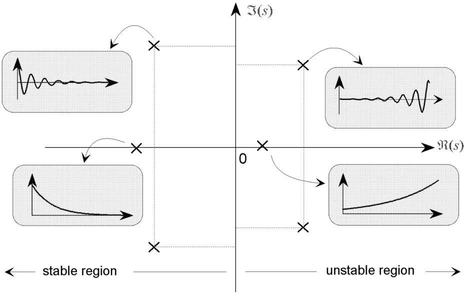
Regulation via state-feedback control

Goal: We want to solve this problem using state feedback $u = -Kx$

$$\dot{x} = A_{cl}x, \quad A_{cl} = (A - BK) \in \mathbb{R}^{n \times n}, K \in \mathbb{R}^{n \times p},$$
$$x(0) = x_0 \neq 0 \in \mathbb{R}^n.$$

Regulation via full state feedback:

- fast with rate $\mu > 0$: place the eigenvalues such that $\mathrm{Re}(\lambda) \leqslant -\mu$
- control over transient: place eigenvalues in certain locations



Location of eignvalues

Regulation via state-feedback control when (A,B) is controllable

Theorem

Let (A, B) be controllable. For every $\alpha > 0$, it is possible to find a state-feedback controller $u = -Kx$ that places all the eigenvalues of the closed-loop matrix $A_{cl} = (A - BK)$ on the complex semi plain Re$[\lambda(A_{cl})] \leqslant -\alpha$.

Design procedure to obtain K such that Re$[\lambda(A_{cl})] \leqslant -\alpha$ for a given $\alpha > 0$:

1. for a given $\alpha > 0$ choose $\mu \geqslant \alpha$ such that $-\mu I - A$ is a stability matrix

2. Because $(-\mu I - A, B)$ is controllable the Lyapunov controllability test says that
$$(-\mu I - A)W + W(-\mu I - A)^\top = -BB^\top,$$
has a unique solution given by $W = \int_0^\infty e^{(-\mu I - A)\tau} BB^\top e^{(-\mu I - A)^\top \tau} d\tau > 0$.

3. Consider the following manipulations
$$(-\mu I - A)W + W(-\mu I - A)^\top = -BB^\top \Leftrightarrow$$
$$PA + A^\top P - PBB^\top P = -2\alpha P, \ (P = W^{-1})$$
Let $K = \frac{1}{2} B^\top P$, then
$$PA + A^\top P - 2PBK = -2\mu P \Leftrightarrow P(A - BK) + (A - BK)^\top P = -2\mu P < 0$$

Regulation via state-feedback control when (A,B) is controllable

Theorem

Let (A, B) be controllable. For every $\alpha > 0$, it is possible to find a state-feedback controller $u = -Kx$ that places all the eigenvalues of the closed-loop matrix $A_{cl} = (A - BK)$ on the complex semi plain Re$[\lambda(A_{cl})] \leqslant -\alpha$.

4. For a $\epsilon > 0$ write $P(A - BK) + (A - BK)^\top P = -2\mu P < 0$ as

$$P(A_{cl} + (\mu - \epsilon)I) + (A_{cl} + (\mu - \epsilon)I)^\top P = -2\epsilon P < 0,$$

Lyapunov stability test: $A_{cl} + (\mu - \epsilon)I$ is a stability matrix,

- $\mathrm{Re}(\lambda(A_{cl} + \mu I - \epsilon)) < 0$.
- $\epsilon \to 0$, we have $\Re(\lambda(A_{cl}) \leqslant -\mu$
- Since $\mu > \alpha$ we can conclude that $\Re(\lambda(A_{cl}) \leqslant -\alpha$.

Conclusion: $K = \frac{1}{2}B^\top W^{-1}$ results in $\mathrm{Re}(\lambda(A_{cl})) \leqslant -\mu \leqslant -\alpha$

$$W^{-1} = \frac{1}{2}B^\top \left(\int_0^\infty e^{(-\mu I - A)\tau} B B^\top e^{(-\mu I - A)^\top \tau} d\tau \right)^{-1}$$

Regulation via state-feedback control when (A,B) is controllable

Theorem (Eigenvalue assignment)

Let (A, B) *be controllable. Given any symmetric set of* n *complex numbers* $\{\nu_1, \nu_2, \cdots, \nu_n\}$*, there exists a full-state feedback matrix* K *such that the closed-loop system matrix* $(A - BK)$ *has eigenvalues equal to these* ν_i*'s.*

see HW 6 for the proof.

Stabilizability for LTI systems

(A, B) uncontrollable: $\text{rank}\, \mathcal{C} = \text{rank}[B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{n-1}B] = m < n$

$$\exists T \text{ (invertible)} : \begin{bmatrix} \dot{x}_c \\ \dot{x}_u \end{bmatrix} = \underbrace{\begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}}_{\bar{A}} \begin{bmatrix} x_c \\ x_u \end{bmatrix} + \underbrace{\begin{bmatrix} B_c \\ 0 \end{bmatrix}}_{\bar{B}} u$$

$$\bar{A} = T^{-1}AT, \quad \bar{B} = T^{-1}B$$

Couple of things here:

- $\lambda[A] = \lambda[\bar{A}]$: because the two systems are allegorically equivalent
- $\lambda[\bar{A}] = \{\lambda[A_c], \lambda[A_u]\}$: because $\bar{A}$ is block triangular
- $\dot{x}_u = A_u x_u$: no controller goes to x_u state equation, eigenvalues of A_u cannot be changed by stat feedback
- (A_c, B_c) is controllable: we can change the eigenvalues of A_c using state feedback

Definition (Stabilizable LTI system)

Def. (Stabilizable system): The pair (A,B) is stabilizable if it is algebraically equivalent to a system in the standard form for uncontrollable systems with $n = m$ (i.e, A_u does not exist) <u>or</u> with A_u a stability matrix.

Definition (Stabilizable LTI system (alternative definition))

The pair (A, B) is stabilizable if there exists a state feedback gain matrix K for which all the eigenvalues of $A - BK$ have strictly negative real part.

Regulation via state-feedback control when (A,B) is controllable: pole-placement/eigenvalue placement

$$\dot{x} = Ax + Bu, \quad u = -Kx \Rightarrow \dot{x} = (A - BK)x$$

- (A, B) controllable: Given any symmetric set of n complex numbers $\{\nu_1, \nu_2, \cdots, \nu_n\}$, there exists a full-state feedback matrix K such that the closed-loop system matrix $(A - BK)$ has eigenvalues equal to these ν_i's.

$$\boxed{\exists K: \ \det(\lambda I - (A - BK)) = \underbrace{(\lambda - \nu_1)(\lambda - \nu_2)\cdots(\lambda - \nu_n)}_{\text{desired charac. polynomial}}}$$

- (A, B) is not controllable: $\operatorname{rank}(\mathcal{C}) = m < n$ ($A \in \mathbb{R}^{n\times n}$):

$$\exists T \text{invertible} : x = T\bar{x} : \quad \dot{\bar{x}} = \begin{bmatrix}\dot{x}_c \\ \dot{x}_u\end{bmatrix} = \underbrace{\begin{bmatrix}A_c & A_{12} \\ 0 & A_u\end{bmatrix}}_{\bar{A} = T^{-1}AT} \begin{bmatrix}x_c \\ x_u\end{bmatrix} + \underbrace{\begin{bmatrix}B_c \\ 0\end{bmatrix}}_{\bar{B} = T^{-1}B} u$$

$$u = -Kx = -KT\bar{x} = -\bar{K}\bar{x} = -\begin{bmatrix}\bar{K}_1 & \bar{K}_2\end{bmatrix}\begin{bmatrix}x_c \\ x_u\end{bmatrix}$$

$$\dot{\bar{x}} = \begin{bmatrix}\dot{x}_c \\ \dot{x}_u\end{bmatrix} = \underbrace{\begin{bmatrix}A_c - B_c\bar{K}_1 & A_{12} - B_c\bar{K}_2 \\ 0 & A_u\end{bmatrix}}_{\bar{A} - \bar{B}\bar{K} = T^{-1}(A - BK)T} \begin{bmatrix}x_c \\ x_u\end{bmatrix}$$

(A_c, B_c) is controllable, we can place eigenvalues of $(A_c - B_c\bar{K}_1)$ in any location we want using state feedback!
We can only change the location of controllable eigenvalues using state feedback

$$\boxed{\text{We can only stabilize a system whose uncontrollable eigenvalues are stable}}$$

$$u = -\bar{K}\bar{x} = -[\bar{K}_1 \ \ 0]\bar{x} = -\underbrace{[\bar{K}_1 \ \ 0]T^{-1}}_{K} x = -K\,x$$

Tests to check stabilizability of LTI systems

There are various stabilizability tests. Following are some of them:

Theorem

The following statements are equivalent:

- *The pair* (A, B) *is stabilizable;*
- *There exists no left eigenvector of A associated with an eigenvalue having nonnegative real part that is orthogonal to the columns of* B*;*

$$\begin{cases} v^\star A = \lambda v & (\textit{Re}[\lambda(A)] \geqslant 0) \\ v^\star B = 0 \end{cases} \implies v = 0$$

- $\text{rank}[\lambda I - A \quad B] = n$ *for all* $\textit{Re}[\lambda(A)] \geqslant 0$.

We have an uncontrollable system (A, B) with $\lambda(A) = \{-2, 3, 0\}$. (notice that $A \in \mathbb{R}^{3\times 3}$)

Q1: Can you find a state feedback gain K such that the eigenvalues of $A_{cl} = (A - BK)$ are $\{-1, -2+3i, -2-3i\}$? A: The answer is no. Here, we want to change the location of all eigenvalues. Because the system is uncontrollable, at least one of the eigenvalues is not controllable, i.e., its location cannot be changed

Q2: When is it feasible to design a state feedback to place the eigenvalues of A_{cl} at $\{3, -2+3i, -2-3i\}$. A: Here, we have changed the location of eigenvalues $\{-2, 0\}$. This can only be possible if these eigenvalues are controllable, that is, for example

$$\text{rank}([-2I - A \ \ B] = 3, \quad \text{rank}([0I - A \ \ B] = 3$$

Q3: Can you find a state feedback gain K which results in $A - BK$ being stable matrix?. A: Because the system is uncontrollable, at least one of the eigenvalues is not controllable, i.e., its location cannot be changed. We can find a state feedback to asymptotically stabilize the system if the only uncontrollable eigenvalue of A is -2. In other words, we should have

$$\text{rank}([3I - A \ \ B] = 3, \quad \text{rank}([0I - A \ \ B] = 3$$

A sample numerical example

Tests to check stabilizability of LTI systems: example

$$\dot{x} = \begin{bmatrix} -11 & 30 \\ -4 & 11 \end{bmatrix} x + \begin{bmatrix} 10 \\ 4 \end{bmatrix} u$$

Controllability text:

$$\text{rank}\, \mathcal{C} = \text{rank}[B \quad AB] = \text{rank} \begin{bmatrix} 10 & 10 \\ 4 & 4 \end{bmatrix} = 1 \Longrightarrow (A, B) \text{ is not controllable!}$$

PBH eigenvalue test for controllability

- first find $\lambda[A]$:
  $\Delta(A) = \det(\lambda I - A) = (\lambda + 11)(\lambda - 11) + 120 = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1) = 0 \Rightarrow \lambda[A] = \{-1, 1\}$
- check rank of $[\lambda I - A \;\; B]$ for $\lambda[A] = \{-1, 1\}$

$$\lambda = -1: \; \text{rank}[-I - A \;\; B] = \text{rank} \begin{bmatrix} 10 & -30 & 10 \\ 4 & -12 & 4 \end{bmatrix} = 1 \Rightarrow \lambda = -1 \text{ is } \underline{\text{not}} \text{ a controllable eigenvalue}$$

$$\lambda = 1: \; \text{rank}[I - A \;\; B] = \text{rank} \begin{bmatrix} 12 & -30 & 10 \\ 4 & -10 & 4 \end{bmatrix} = 2 \Rightarrow \lambda = 1 \text{ is a controllable eigenvalue}$$

Then:

- (A,B) is not controllable
- (A,B) is stablilizable: because $\text{rank}[\lambda I - A \;\; B] = n$ for the eigenvalue with positive real part $\lambda = 1$.

Feedback controller design for stabilizable LTI systems: example

Objective Place eigenvalues of $A - BK$ at $\{-1, -3\}$

Consider the state feedback $u = -Kx = -[k_1\ k_2]x$

$$A_{cl} = A - BK = \begin{bmatrix} -11 - 10k_1 & 30 - 10k_2 \\ -4 - 4k_1 & 11 - 4k_2 \end{bmatrix}$$

$$\Delta(A_c l) = \Delta(A - BK) = \det\left(\lambda I - \begin{bmatrix} -11 - 10k_1 & 30 - 10k_2 \\ -4 - 4k_1 & 11 - 4k_2 \end{bmatrix}\right) = (\lambda + 1)(\lambda + 10k_1 + 4k_2 - 1) = 0$$

$$\lambda[A_{cl}] = \{-1, -10k_1 - 4k_2 + 1\}$$

Notice that we cannot change the location of uncontrollable eigenvalue but we can put the controllable eigenvalue in any new location using state feedback!

We can pick k_1 and k_2 such that A_{cl} has eigenvalues with strictly negative real parts and, as such, stabilize the closed-loop system using $u = -Kx$.

For example $k_1 = 0$ and $k_2 = 1$ results in $\lambda[A_{cl}] = \{-1, -3\}$.

You can confirm this by checking eigenvalues of $A - BK = \begin{bmatrix} -11 & 30 \\ -4 & 11 \end{bmatrix} - \begin{bmatrix} 10 \\ 4 \end{bmatrix} [0\ \ 1] = \begin{bmatrix} -11 & 20 \\ -4 & 7 \end{bmatrix}$.

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See the next slide for an alternative design approach:

- Note: state feedback gain that places the eigenvalues in certain locations is not necessarily unique

Feedback controller design for stabilizable LTI systems: example

$$\dot{x} = \begin{bmatrix} -11 & 30 \\ -4 & 11 \end{bmatrix} x + \begin{bmatrix} 10 \\ 4 \end{bmatrix} u$$

$\lambda[A]$: $\Delta(A) = \det(\lambda I - A) = (\lambda + 11)(\lambda - 11) + 120 = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1) = 0 \Rightarrow \lambda[A] = \{-1, 1\}$

Objective Place eigenvalues of $A - BK$ at $\{-1, -3\}$

Controllability text:

$$\text{rank}\, \mathcal{C} = \text{rank}[B \quad AB] = \text{rank} \begin{bmatrix} 10 & 10 \\ 4 & 4 \end{bmatrix} = 1 \Longrightarrow (A, B) \text{ is not controllable!}$$

Controllable decomposition

$$T = \begin{bmatrix} 5 & 0 \\ 2 & 0.2 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 0.2 & 0 \\ -2 & 5 \end{bmatrix}$$

$$\bar{A} = T^{-1} A T = \begin{bmatrix} 0.2 & 0 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} -11 & 30 \\ -4 & 11 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 2 & 0.2 \end{bmatrix} = \begin{bmatrix} 1 & -1.2 \\ 0 & -1 \end{bmatrix}, \quad \bar{B} = T^{-1} B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

We use $(A_c, B_c) = (1, 2)$ to place eigenvalue of the controllable part at -3: $\lambda(A_c - B_c \bar{k}_1) = -3$

$$\lambda - (1 - 2\bar{k}_1) = \lambda + 3 \;\Rightarrow \bar{k}_1 = 2.$$

$$u = \bar{K}\bar{x} = -\begin{bmatrix} 2 & 0 \end{bmatrix} \bar{x} = -\begin{bmatrix} 2 & 0 \end{bmatrix} T^{-1} x = -\begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} 0.2 & 0 \\ -2 & 5 \end{bmatrix} x = -\underbrace{\begin{bmatrix} 0.4 & 0 \end{bmatrix}}_{K} x.$$

You can confirm this by checking eigenvalues of $A - BK = \begin{bmatrix} -11 & 30 \\ -4 & 11 \end{bmatrix} - \begin{bmatrix} 10 \\ 4 \end{bmatrix} [0.4 \;\; 0] = \begin{bmatrix} -15 & 30 \\ -5.6 & 11 \end{bmatrix}.$