# Linear Systems I
# Lecture 12

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Complementary Reading: Ch 6.1, 6.2 and 6.8 from Ref[1].

Note: These slides only cover part of the discussions in the class. For further details, consult your in-class notes.

# This lecture: controllability and reachability concepts for LTI systems

- Controllable and reachable subspaces

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du, \end{cases} \qquad x(t_0) = x_0 \in \mathbb{R}^n$$

  - Can we steer the system states from zero initial conditions to any place in the space in finite time? If not, what are such points?
  - Can we steer the system states from any arbitrary point in the space to the origin in finite time? If not, what are such points?

# Review of Controllable and reachable subspaces for LTV systems

$$\begin{cases} \dot{x} = A(t)x + B(t)u, \\ y = C(t)x + D(t)u, \end{cases} \qquad x(t_0) = x_0 \in \mathbb{R}^n$$

$$x(t) = \phi(t, t_0)x_0 + \int_{t_0}^{t} \phi(t, \tau)B(\tau)u(\tau)d\tau \Rightarrow$$

$$\text{at } t = t_1: \quad x_1 = x(t_1) = \phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

## Definition (Reachable subspace (controllable-from-the-origin))

Given two times $t_1 > t_0 \geqslant 0$, starting from $x_0 = 0$,

$$\mathcal{R}[t_0, t_1] := \left\{ x_1 \in \mathbb{R}^n : \exists u(.), x_1 = \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)u(\tau)d\tau \right\}$$

## Definition (Controllable subspace (controllable-to-the-origin))

Given two times $t_1 > t_0 \geqslant 0$, starting from $x_0 \neq 0$,

$$\mathcal{C}[t_0, t_1] := \left\{ x_0 \in \mathbb{R}^n : \exists u(.), 0 = \phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)u(\tau)d\tau \right\}$$

$$\mathcal{C}[t_0, t_1] := \left\{ x_0 \in \mathbb{R}^n : \exists v(.) = -u(.), x_0 = \int_{t_0}^{t_1} \phi(t_0, \tau)B(\tau)v(\tau)d\tau \right\}$$

# Review of Reachability gramians for LTV systems

**Definition (Reachability and controllability gramians for given $t_1 > t_0 \geqslant 0$)**

$$\text{Reachability gramian:} \quad W_R(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)B(\tau)^\top \phi(t_1, \tau)^\top d\tau,$$

**Theorem (Reachable subspace)**

*Given two times* $t_1 > t_0 \geqslant 0$,

$$\mathcal{R}[t_0, t_1] = \mathit{Im} W_R(t_0, t_1),$$

*Moreover, if* $x_1 = W_R(t_0, t_1)\eta_1 \in \mathit{Im} W_R(t_0, t_1)$, *the control*

$$u(t) = B(t)^\top \phi(t_1, t)^\top \eta_1, \qquad t \in [t_0, t_1], \qquad \textit{minimum-energy open-loop controller}$$

*can be used to transfer the state from* $x(t_0) = 0$ *to* $x(t_1) = x_1$.

# Review of Controllability gramians for LTV systems

**Definition (Reachability and controllability gramians for given $t_1 > t_0 \geqslant 0$)**

$$\text{Controllability gramian:} \quad W_C(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_0, \tau) B(\tau) B(\tau)^\top \phi(t_0, \tau)^\top d\tau,$$

**Theorem (Controllable subspace)**

*Given two times* $t_1 > t_0 \geqslant 0$,

$$\mathcal{C}[t_0, t_1] = \mathit{Im} W_C(t_0, t_1),$$

*Moreover, if* $x_0 = W_C(t_0, t_1)\eta_0 \in \mathit{Im} W_C(t_0, t_1)$, *the control*

$$u(t) = -B(t)^\top \phi(t_0, t)^\top \eta_0, \qquad t \in [t_0, t_1], \qquad \textit{minimum-energy open-loop controller}$$

*can be used to transfer the state from* $x(t_0) = x_0$ *to* $x(t_1) = 0$.

# Completely-state controllable and reachable LTV systems

$$\dot{x} = A(t)x + B(t)u, \quad x \in \mathbb{R}^n$$

**Definition ((Completely-state) reachable system)**

Given two times $t_1 > t_0 \geqslant 0$, starting from $x_0 = 0$,

$$\left\{x_1 \in \mathbb{R}^n : \exists u(.), x_1 = \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)u(\tau)d\tau\right\} = \mathbb{R}^n$$

How to check: $\mathrm{rank}(W_R(t_0, t_1)) = n$

**Definition ((Completely-state) controllable system)**

Given two times $t_1 > t_0 \geqslant 0$, starting from $x_0 \neq 0$,

$$\left\{x_0 \in \mathbb{R}^n : \exists v(.) = -u(.), x_0 = \int_{t_0}^{t_1} \phi(t_0, \tau)B(\tau)v(\tau)d\tau\right\} = \mathbb{R}^n$$

How to check: $\mathrm{rank}(W_R(t_0, t_1)) = n$

# Controllable and reachable subspaces for LTI systems

$$\begin{cases} \dot{x} = A\,x + B\,u, \\ y = C\,x + D\,u, \end{cases} \qquad x(t_0) = x_0 \in \mathbb{R}^n$$

$$x(t) = \phi(t,t_0)x_0 + \int_{t_0}^{t} \phi(t,\tau)B(\tau)u(\tau)d\tau = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}B(\tau)u(\tau)d\tau \Rightarrow$$

$$x_1 = x(t_1) = \phi(t_1,t_0)x_0 + \int_{t_0}^{t_1} \phi(t_1,\tau)B(\tau)u(\tau)d\tau = e^{A(t_1-t_0)}x_0 + \int_{t_0}^{t_1} e^{A(t_1-\tau)}B(\tau)u(\tau)d\tau$$

**Definition (Reachable subspace (controllable-from-the-origin))**

Given two times $t_1 > t_0 \geqslant 0$, starting from $x_0 = 0$,

$$\mathcal{R}[t_0,t_1] := \left\{ x_1 \in \mathbb{R}^n : \exists u(.), x_1 = \int_{t_0}^{t_1} \phi(t_1,\tau)B(\tau)u(\tau)d\tau = \int_{t_0}^{t_1} e^{A(t_1-\tau)}B(\tau)u(\tau)d\tau \right\}$$

**Definition (Controllable subspace (controllable-to-the-origin))**

Given two times $t_1 > t_0 \geqslant 0$, starting from $x_0 \neq 0$,

$$\mathcal{C}[t_0,t_1] := \left\{ x_0 \in \mathbb{R}^n : \exists u(.), 0 = \phi(t_1,t_0)x_0 + \int_{t_0}^{t_1} \phi(t_1,\tau)B(\tau)u(\tau)d\tau \right.$$

$$\left. = e^{A(t_1-t_0)}x_0 + \int_{t_0}^{t_1} e^{A(t_1-\tau)}B(\tau)u(\tau)d\tau \right\}$$

$$\mathcal{C}[t_0,t_1] := \left\{ x_0 \in \mathbb{R}^n : \exists v(.) = -u(.), x_0 = \int_{t_0}^{t_1} e^{A(t_0-\tau)}B(\tau)v(\tau)d\tau \right\}$$

# Controllability and reachability gramians for LTI systems

$$\dot{x} = Ax + Bu, \quad x(t_0) = x_0 \in \mathbb{R}^n$$

**Definition (Reachability and controllability gramians for given $t_1 > t_0 \geqslant 0$)**

$$W_R(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)B(\tau)^\top \phi(t_1, \tau)^\top d\tau = \int_{t_0}^{t_1} e^{A(t_1-\tau)}BB^\top e^{A^\top(t_1-\tau)} d\tau.$$

$$W_C(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_0, \tau)B(\tau)B(\tau)^\top \phi(t_0, \tau)^\top d\tau = \int_{t_0}^{t_1} e^{A(t_0-\tau)}BB^\top e^{A^\top(t_0-\tau)} d\tau.$$

Alternatively, we can write

**Definition (Reachability and controllability gramians over any finite time interval $[0, T]$)**

$$W_R(t_0, t_1) = \int_0^{t_1-t_0} e^{A\tau}BB^\top e^{A^\top(\tau)} d\tau = \int_0^{T} e^{At}BB^\top e^{A^\top t} dt.$$

$$W_C(t_0, t_1) = \int_0^{t_1-t_0} e^{-A\tau}BB^\top e^{-A^\top \tau} d\tau = \int_0^{T} e^{-At}BB^\top e^{-A^\top t} dt.$$

# Controllability matrix for LTI systems

$$\dot{x} = Ax + Bu, \quad x(t_0) = x_0 \in \mathbb{R}^n$$

**Definition (Reachability and controllability gramians for over any finite time interval $[0, T]$)**

$$W_R(t_0, t_1) = \int_0^{t_1 - t_0} e^{A\tau} BB^\top e^{A^\top(\tau)} d\tau = \int_0^T e^{At} BB^\top e^{A^\top t} dt.$$

$$W_C(t_0, t_1) = \int_0^{t_1 - t_0} e^{-A\tau} BB^\top e^{-A^\top \tau} d\tau = \int_0^T e^{-At} BB^\top e^{-A^\top t} dt.$$

**Theorem**

*Let*

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}_{n \times (np)}.$$

*For any two time* $t_1 > t_0 \geqslant 0$

$$\mathcal{R}[t_0, t_1] = \mathit{Im} W_R(t_0, t_1) = \mathit{Im} \mathcal{C} = \mathit{Im} W_C(t_0, t_1) = \mathcal{C}[t_0, t_1].$$

# Controllable and reachable subspaces: example

$$\dot{x} = \begin{bmatrix} -\frac{1}{R_1C_1} & 0 \\ 0 & -\frac{1}{R_2C_2} \end{bmatrix} x + \begin{bmatrix} \frac{1}{R_1C_1} \\ \frac{1}{R_2C_2} \end{bmatrix} u$$

This is an LTI system, therefore the controllable and reachable subsets are equal to one and other and can be obtained from finding Image (range) of controllability matrix:

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- $\omega = \frac{1}{R_1C_1} = \frac{1}{R_2C_2}$

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} \omega & -\omega^2 \\ \omega & -\omega^2 \end{bmatrix}$$

$\mathcal{C}$ has one linearly independent column. The reachable and controllable subsets are ($\alpha \in \mathbb{R}$):

$$\mathrm{Im}C = \alpha \begin{bmatrix} \omega \\ \omega \end{bmatrix} = \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathcal{R}(t_0, t_1) = C(t_0, t_1)$$

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- $\omega_1 = \frac{1}{R_1C_1} \neq \omega_2 = \frac{1}{R_2C_2}$

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} \omega_1 & -\omega_1^2 \\ \omega_2 & -\omega_2^2 \end{bmatrix}$$

$\mathcal{C}$ has two linearly independent columns. The reachable and controllable subsets are ($\alpha, \beta \in \mathbb{R}$):

$$\mathrm{Im}C = \alpha \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} + \beta \begin{bmatrix} \omega_1^2 \\ \omega_2^2 \end{bmatrix} = \mathbb{R}^2 = \mathcal{R}(t_0, t_1) = C(t_0, t_1)$$

In this case every point in the $\mathbb{R}^2$ is reachable from the origin in <u>finite time</u> and every point in the $\mathbb{R}^2$ can be steered to origin in <u>finite time</u>.

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# Controllable LTI systems

## Definition

The state equation $\dot{x} = Ax + Bu$ or the pair $(A, B)$ is said to be controllable if for any initial state $x(0) = x_0$ and any final state $x_1$, there exists an input that transfers $x_0$ to $x_1$ in a finite time. Otherwise $(A, B)$ is said to be uncontrollable.

## Theorem

*Let*

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}_{n\times(np)}.$$

*For any two time* $t_1 > t_0 \geqslant 0$

$$\mathcal{R}[t_0, t_1] = ImW_R(t_0, t_1) = Im\mathcal{C} = ImW_C(t_0, t_1) = \mathcal{C}[t_0, t_1].$$

## Theorem

*The following statements are equivalent:*

1. *The pair $(A, B)$ is controllable.*
2. *The matrix below is nonsingular for any $t > 0$*

$$W_C(t) = \int_0^t e^{At}BB^\top e^{A^\top t}dt$$

3. *The $n \times (np)$ controllability matrix $\mathcal{C}$ is full row rank*

$$Rank\,\mathcal{C} = Rank\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}_{n\times(np)} = n.$$

# Rest of today's lecture

- Controllability of LTI systems

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du, \end{cases} \qquad x(t_0) = x_0 \in \mathbb{R}^n$$

  - Can we steer the system states from every point in $\mathbb{R}^n$ to every other point in $\mathbb{R}^n$ in finite time? ((completely-state) controllable system)
    - test to evaluate controllability

# Controllable decomposition

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^p$$

## Theorem

$$\text{rank} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = m < n$$

$\exists\, T$ *invertible s.t.* $x = T\bar{x}$ *transforms state equations to*

$$\bar{A} = T^{-1}AT = \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B_c \\ 0 \end{bmatrix}$$

$$A_c \in \mathbb{R}^{m \times m}, \quad , B_c \in \mathbb{R}^{m \times p}, \quad A_u \in \mathbb{R}^{(n-m) \times (n-m)}, \quad A_{12} \in \mathbb{R}^{m \times (n-m)}.$$

## Corollary

- *The pair* $(A_c, B_c)$ *is controllale, i.e.,* $\text{rank} \begin{bmatrix} B_c & A_cB_c & \cdots & A_c^{m-1}B_c \end{bmatrix} = m$
- *The controllable subspace of* $(\bar{A}, \bar{C})$ *is Im* $\begin{bmatrix} I_{m \times m} \\ 0_{(n-m) \times m} \end{bmatrix}$

# Controllable decomposition: example

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} u$$

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 1 & -3 \\ 1 & -3 & 7 \\ -3 & 7 & -15 \end{bmatrix}$$

$\mathcal{C}$ has only two linearly independent columns: $A^2B = -2\,B - 3\,AB$

## Controllable decomposition

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -3 & 0 \\ -3 & 7 & 1 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

$$\bar{A} = T^{-1}AT = \left[\begin{array}{cc|c} 0 & -2 & 1 \\ 1 & -3 & 0 \\ \hline 0 & 0 & -3 \end{array}\right], \quad \bar{B} = T^{-1}B = \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \end{array}\right]$$

$$A_c = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

# Controllable decomposition: transfer function

**Theorem**

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^p$$
$$y = Cx + D$$

$$\text{rank}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = m < n$$

$\exists\, T$ *invertible s.t.* $x = T\bar{x}$ *transforms state equations to*

$$\bar{A} = T^{-1}AT = \begin{bmatrix} A_c & A_{12} \\ 0_{(n-m)\times m} & A_u \end{bmatrix}, \quad \bar{B} = T^{-1}B = \begin{bmatrix} B_c \\ 0_{(n-m)\times p} \end{bmatrix}$$
$$\bar{C} = CT = \begin{bmatrix} C_c & C_u \end{bmatrix}, \quad \bar{D} = D$$

For (A,B,C,D): $\hat{G}(s) = C(sI - A)^{-1}B + D$.

Transfer function of two algebraically equivalent system is the same

$$\hat{G}(s) = \hat{\bar{G}}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + D = \begin{bmatrix} C_c & C_u \end{bmatrix} \begin{bmatrix} (sI - A_c) & -A_{12} \\ 0 & (sI - A_u) \end{bmatrix}^{-1} \begin{bmatrix} B_c \\ 0 \end{bmatrix} + D$$

$$= \begin{bmatrix} C_c & C_u \end{bmatrix} \begin{bmatrix} (sI - A_c)^{-1} & \times \\ 0 & (sI - A_u)^{-1} \end{bmatrix} \begin{bmatrix} B_c \\ 0 \end{bmatrix} + D = C_c(sI - A_c)^{-1}B_c + D.$$

$$\hat{G}(s) = C_c(sI - A_c)^{-1}B_c + D$$

Transfer function of an LTI system is equal to the transfer function of its controllable part.

# Controllable decomposition: example

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x$$

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 1 & -3 \\ 1 & -3 & 7 \\ -3 & 7 & -15 \end{bmatrix}$$

$\mathcal{C}$ has only two linearly independent columns: $A^2B = -2\,B - 3\,AB$

Controllable decomposition

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -3 & 0 \\ -3 & 7 & 1 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

$$\bar{A} = T^{-1}AT = \left[\begin{array}{cc|c} 0 & -2 & 1 \\ 1 & -3 & 0 \\ \hline 0 & 0 & -3 \end{array}\right], \quad \bar{B} = T^{-1}B = \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \end{array}\right], \quad \bar{C} = CT = \left[\begin{array}{cc|c} 1 & -2 & 0 \end{array}\right]$$

$$A_c = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\hat{G}(s) = C_c(sI - A_c)^{-1}B_c + D = \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} s & 2 \\ -1 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{s-2}{(s+1)(s+2)}$$