Linear Systems I
Lecture 10

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Reading: Ch 5.1-5.3, Ch 5.5, Ch 6.1, Example 6.3 of Ref [1] (skip over the parts that cover discrete-time systems).

Note: These slides only cover part of the discussions in the class. For further details, consult your in-class notes.

This lecture

- Stability
  - Internal stability of LTI systems (Lyapunov method)
  - Bounded-Input-Bounded-Output (BIBO) stability
- Controllable and Reachable subspaces

Stability of LTV/LTI systems: BIBO stability

$$\begin{cases} \dot{x} = A(t)x + B(t)u, \\ y = C(t)x + D(t)u, \end{cases} \qquad x(t_0) = 0_n$$

Stability addresses what happens to our solutions

- do they remain bounded
- will they get progressively smaller
- they diverge to infinity

Response is due to : $\underbrace{\text{response due to } x_0}_{\text{internal stability}} + \underbrace{\text{response due to } u}_{\text{Input-output stability}}$

Def(Bounded-input-bounded-output (BIBO) stability): A system is said to be BIBO stable if every bounded input excites a bounded output (for BIBO stability evaluation we analyze the zero-state response of the system).

An input $u(t)$ is **said to be bounded** if $u(t)$ does not grow to positive or negative infinity, or equivalently, $\exists$ a constant u_m s.t.

$$|u(t)| \leqslant u_m < \infty, \quad \forall t \geqslant 0.$$

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du, \end{cases} \qquad x(t_0) = 0_n \qquad (\star)$$

$$y_f(t) = y_{zs}(t) = \int_0^t g(t-\tau)u(\tau)d\tau = \int_0^t g(\tau)u(t-\tau)d\tau$$

Theorem

A SISO system $(\star)$ is BIBO if and only if $g(t)$ is absolutely integrable in $[0, \infty)$ or

$$\int_0^\infty |g(t)|dt \leqslant M < \infty$$

BIBO stability of SISO LTI systems

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du, \end{cases} \qquad x(t_0) = 0_n \qquad (\star)$$

$$y_f(t) = y_{zs}(t) = \int_0^t Ce^{A\,(t-\tau)}Bu(\tau)d\tau + Du(t), \qquad \bar{g}(t) := Ce^{A\,t}B$$

$$y_f(t) = y_{zs}(t) = \int_0^t \bar{g}(t-\tau)u(\tau)d\tau + Du(t) = \int_0^t \bar{g}(\tau)u(t-\tau)d\tau + Du(t)$$

Corollary

A SISO system $(\star)$ *is BIBO if and only if* $\bar{g}(t) = Ce^{A\,t}B$ *is absolutely integrable in* $[0, \infty)$ *or*

$$\int_0^\infty |\bar{g}(t)|dt \leqslant M < \infty$$

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du, \end{cases} \qquad x(t_0) = x_0 \in \mathbb{R}^n \ , \bar{g}(t) := Ce^{A\,t}B$$

$$y_f(t) = y_{zs}(t) = \int_0^t \bar{g}(t-\tau)u(\tau)d\tau + Du(t) = \int_0^t \bar{g}(\tau)u(t-\tau)d\tau + Du(t)$$

$$\hat{\bar{g}}(s) = \mathcal{L}[Ce^{A\,t}B] = C(sI-A)^{-1}B$$

$$\begin{aligned} \hat{\bar{g}}(s) =& \frac{\alpha_1 s^{\nu-1} + \alpha_2 s^{\nu-2} + \cdots + \alpha_\nu}{(s-\lambda_1)^{m_1}(s-\lambda_2)^{m_2}\cdots(s-\lambda_k)^{m_k}} \\ =& \frac{a_{11}}{s-\lambda_1} + \frac{a_{12}}{(s-\lambda_1)^2} + \cdots + \frac{a_{1m_1}}{(s-\lambda_1)^{m_1}} + \cdots \\ & + \frac{a_{k1}}{s-\lambda_k} + \frac{a_{k2}}{(s-\lambda_k)^2} + \cdots + \frac{a_{km_k}}{(s-\lambda_k)^{m_k}} \end{aligned}$$

$$\begin{aligned} \bar{g}(t) = \mathcal{L}^{-1}[\hat{\bar{g}}(s)] =& \bar{a}_{11}e^{\lambda_1 t} + \bar{a}_{12}te^{\lambda_1 t} + \cdots + \bar{a}_{1m_1}t^{m_1-1}e^{\lambda_1 t} + \cdots + \\ & \bar{a}_{k1}e^{\lambda_k t} + \bar{a}_{k2}te^{\lambda_k t} + \cdots + \bar{a}_{km_k}t^{m_k-1}e^{\lambda_k t} \end{aligned}$$

BIBO stability of SISO LTI systems

$$\bar{g}(t) = \mathcal{L}^{-1}[\hat{\bar{g}}(s)] = \bar{a}_{11}e^{\lambda_1 t} + \bar{a}_{12}te^{\lambda_1 t} + \cdots + \bar{a}_{1m_1}t^{m_1-1}e^{\lambda_1 t} + \cdots + \bar{a}_{k1}e^{\lambda_k t} + \bar{a}_{k2}te^{\lambda_k t} + \cdots + \bar{a}_{km_k}t^{m_k-1}e^{\lambda_k t}$$

Notice that some of $\bar{a}_{ij}$'s can be zero.

Corollary

A SISO system ($\star$) is BIBO if and only if $\bar{g}(t) = Ce^{At}B$ is absolutely integrable in $[0, \infty)$ or

$$\int_0^\infty |\bar{g}(t)|dt \leqslant M < \infty$$

1. If all the poles p_l of $\hat{\bar{g}}(s)$ have strictly negative real parts, then $\bar{g}(t) \to 0$ as $t \to \infty$. Then the system is BIBO stable.
2. If at least one of the $\hat{\bar{g}}(s)$'s poles has a pole p_l with zero or positive real part, then $|\bar{g}(t)|$ does not converge to zero and the system is **not** BIBO stable.

Theorem (Frequency domain BIBO condition)

The following statements are equivalent

1. *The LTI system ($\star$) is uniformly BIBO stable.*
2. *Every pole of the transfer function of the system has strictly negative real parts.*

BIBO stability of MIMO LTI systems

Theorem (Time domain BIBO condition)

A MIMO LTI system with impulse response matrix $G(t) = [g_{ij}(t)]$ *is BIBO stable, if and only if every* $g_{ij}(t)$ *is absolutely integrable in* $[0, \infty)$.

Theorem (Frequency domain BIBO condition)

A MIMO LTI system with proper rational transfer function $\hat{G}(s) = [\hat{g}_{ij}] = [\hat{\bar{g}}_{ij}] + D$ *is BIBO stable iff every pole of* $\hat{\bar{g}}_{ij}$ *has negative real parts.*

Corollary

*When the LTI system is exponentially stable, then it must be also BIBO stable. The converse of this statement is **not** true!*

Recall that, for a system $\{A, B, C, D\}$, the poles of its transfer function are $\subseteq$ of set of eigenvalues of system matrix A

The following two systems are internally unstable but they are BIBO stable

- $\dot{x} = \begin{bmatrix} -2 & 5 \\ 0 & 3 \end{bmatrix} x + \begin{bmatrix} 4 \\ 0 \end{bmatrix} u,\ y = \begin{bmatrix} 7 & 8 \end{bmatrix} x + 1.5u$: eigenvalues $= \{-2, 3\}$, poles of transfer function $= \{-2\}$
- $\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,\ y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$: eigenvalues $= \{1, -2\}$, poles of transfer function $= \{-2\}$

BIBO stability of LTV systems

$$\begin{cases} \dot{x} = A(t)x + B(t)u, \\ y = C(t)x + D(t)u, \end{cases} \quad , \quad (\star)$$

$$\bar{G}(t,\tau) := C(\tau)\phi(t,\tau)B(\tau) = [\bar{g}_{ij}(t,\tau)]$$

$$y_f(t) = y_{zs}(t) = \int_0^t C(\tau)\phi(t,\tau)B(\tau)u(\tau)d\tau + D(t)u(t) =$$
$$\int_0^t \bar{G}(t,\tau)u(\tau)d\tau + D(t)u(t) = \int_0^t \bar{G}(\tau)u(t,\tau)d\tau + D(t)u(t)$$

Theorem (Time domain BIBO stability condition for LTV)

The following statements are equivalent

1. *The LTV system* $(\star)$ *is uniformly BIBO stable.*
2. *Every every of* $D(.)$ *is uniformly bounded and*

$$\int_0^t |\bar{g}_{ij}(t,\tau)|d\tau \leqslant M_{ij} \leqslant \infty.$$

Rest of this lecture

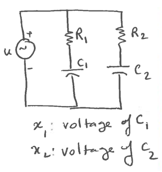
- Controllable and reachable subspaces

$$\begin{cases} \dot{x} = A(t)x + B(t)u, \\ y = C(t)x + D(t)u, \end{cases} \qquad x(t_0) = x_0 \in \mathbb{R}^n$$

  - Can we steer the system states from zero initial conditions to any place in the space in finite time? If not every point, what subset of points we can go to?
  - Can we steer the system states from any arbitrary point in the space to the origin in finite time? If not every point, what subset of points we can steer to origion?

Controllable and reachable subspaces: example

$$\dot{x} = \begin{bmatrix} -\frac{1}{R_1 C_1} & 0 \\ 0 & -\frac{1}{R_2 C_2} \end{bmatrix} x + \begin{bmatrix} \frac{1}{R_1 C_1} \\ \frac{1}{R_2 C_2} \end{bmatrix} u$$

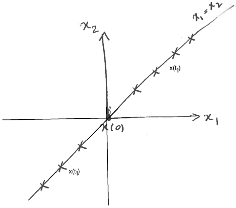


$$x(t) = \begin{bmatrix} e^{-\frac{t}{R_1 C_1}} x_1(0) \\ e^{-\frac{t}{R_2 C_2}} x_2(0) \end{bmatrix} + \int_0^t \begin{bmatrix} \frac{e^{-\frac{t-\tau}{R_1 C_1}}}{R_1 C_1} \\ \frac{e^{-\frac{t-\tau}{R_2 C_2}}}{R_2 C_2} \end{bmatrix} u(\tau) d\tau$$

when $R_1 C_1 = R_2 C_2 = 1/\omega$: $x(t) = e^{-\omega t} x(0) + \omega \int_0^t e^{-\omega(t-\tau)} u(\tau) d\tau \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Starting from $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, the question here is what points in the space are reachable in finite time $t_1 > 0$ by applying a control input.

$$x(t) = e^{-\omega t} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \omega \int_0^t e^{-\omega(t-\tau)} u(\tau) d\tau \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \underbrace{\omega \int_0^t e^{-\omega(t-\tau)} u(\tau) d\tau}_{\alpha(t)} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

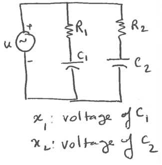


$$x(t_1) = \alpha(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \alpha(t_1) := \omega \int_0^{t_1} e^{-\omega(t-\tau)} u(\tau) d\tau$$

reachable points: $x(t_1) = \left\{ \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} : \alpha \in \mathbb{R} \right\}, \quad \forall t_1 > t_0 \geqslant 0.$

Controllable and reachable subspaces: example

$$\dot{x} = \begin{bmatrix} -\frac{1}{R_1 C_1} & 0 \\ 0 & -\frac{1}{R_2 C_2} \end{bmatrix} x + \begin{bmatrix} \frac{1}{R_1 C_1} \\ \frac{1}{R_2 C_2} \end{bmatrix} u$$

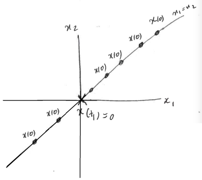


$$x(t) = \begin{bmatrix} e^{-\frac{t}{R_1 C_1}} x_1(0) \\ e^{-\frac{t}{R_2 C_2}} x_2(0) \end{bmatrix} + \int_0^t \begin{bmatrix} \frac{e^{-\frac{t-\tau}{R_1 C_1}}}{R_1 C_1} \\ \frac{e^{-\frac{t-\tau}{R_2 C_2}}}{R_2 C_2} \end{bmatrix} u(\tau) d\tau$$

when $R_1 C_1 = R_2 C_2 = 1/\omega$: $x(t) = e^{-\omega t} x(0) + \omega \int_0^t e^{-\omega(t-\tau)} u(\tau) d\tau \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The question here is what initial conditions $x(0) = x_0 \in \mathbb{R}^2$ can be steered to origin in finite time $t_1 > 0$ ($x(t_1) = 0$) using control inputs. Therefore, our objective is to find x_0 that satisfy

$$0 = e^{-\omega t} x(0) + \omega \int_0^{t_1} e^{-\omega(t-\tau)} u(\tau) d\tau \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$



As you can see, our objective is only achievable $x(0)$ is aligned with $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, note that

$$x(0) = \underbrace{-\omega e^{\omega t} \int_0^{t_1} e^{-\omega(t-\tau)} u(\tau) d\tau}_{\beta(t_1)} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Therefore, the 'controllable' initial conditions are $x_0 = \left\{ \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix} : \beta \in \mathbb{R} \right\}, \quad \forall t_1 > t_0 \geqslant 0.$