HOMEWORK 7
MAE 206- OPTIMIZATION METHODS
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Problem 1. Consider the optimization problem below from your HW 5

$$\text{minimize } f(x) = x_1 + x_2^2 + x_2\, x_3 + 2x_3^2, \quad \text{subject to,}$$
$$\frac{1}{2}(x_1^2 + x_2^2 + x_3^2) = 1.$$

Use the code that is published with this HW to solve this problem numerically using the practical quadratic penalty function algorithm. Use the initial conditions $x_0 = \begin{bmatrix} 1 \\ 0.55 \\ -0.2 \end{bmatrix}$ and $x_0 = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$. Modify the code to plot the approximate solution of each penalty function.

Problem 2. Modify the code for the practical penalty method you use in Problem 1 to solve Problem 1 using Augmented Lagrangian method.

Problem 3. *Exact penalty function's advantage over the quadratic penalty function is that the penalty function parameter c can be a finite value, so you do not need to solve an infinite sequence of penalty problems to obtain the correct solution. However, the success of this algorithm depends on the boundedness of the Lagrange multipliers. The following theorem gives the necessary and sufficient conditions that guarantees the boundedness of the Lagrange multipliers.*

Theorem: Let $x^\star$ be a local minimum of f subject to $h_i(x)$, $i = 1, \cdots, m$, $g_j(x) \leq 0$, $j = 1, \cdots, r$, where f, h_i and g_j are continuously differentiable. Consider the set of pairs $(\lambda^\star, \mu^\star)$ that are Lagrange multipliers, i.e., satisfy $\mu_j^\star \geq 0, \mu_j^\star g_j(x^\star) = 0$ for all j, and $\nabla L(x^\star, \lambda^\star, \mu^\star) = 0$. This set is nonempty and bounded if and only if the Mangasarian-Fromovitz constraint qualification (MFCQ) holds.

MFCQ: Let $x^\star$ be a local minimum of f subject to $h_i(x)$, $i = 1, \cdots, m$, $g_j(x) \leq 0$, $j = 1, \cdots, r$, where f, h_i and g_j are continuously differentiable. We say that the MGCQ holds at $x^\star$ if the gradients $\nabla h_i(x^\star)$, $i = 1, \cdots, m$ are linearly independent, and that there exists a vector $d \in \mathbb{R}^n$ such that

$$\nabla h_i(x^\star)^\top d = 0, \; i = 1, \cdots, m, \quad \nabla g_j(x^\star)^\top d < 0, \; \forall j \in A(x^\star).$$

Consider the problem

$$\text{minimize } f(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2), \quad \text{subject to,}$$
$$0.5x_1 - 0.4x_2 + x_3 = 0.7,$$
$$1 \leq x_i \leq 2, \; j = 1, 2, 3.$$

This problem is a convex optimization problem and its minimizers are guaranteed to be captured by the KKT equations.

(a) Show that $x^\star = (1, 2, 1)$ is a minimizer of the problem. You do this by showing that there exists $\lambda^\star \in \mathbb{R}^3$ and $\mu^\star \in \mathbb{R}^3_{\geq 0}$ such that $\nabla_x L(x^\star, \lambda^\star, \mu^\star) = 0$ along with showing that $x^\star$ satisfies the constraints. To set up your Lagrangian, think about which inequality constraints are active, so you can obtain the right $g_j(x) \leq 0$ forms.

(b) Are the Lagrange multipliers you found in part (a) unique? Are they bounded? Confirm your observation by checking the MFCQ condition.

(c) Will you recommend using exact penalty function to solve this problem numerically?