Controllers for Linear Systems with Bounded Actuators: Slab Scheduling and Anti-windup *

Solmaz Sajjadi-Kia [a], Faryar Jabbari [b],

[a] *Department of Mechanical and Aerospace Engineering, University of California San Diego, San Diego, CA 92093, USA*

[b] *Department of Mechanical and Aerospace Engineering, University of California Irvine, Irvine, CA 92697, USA*

Abstract

For linear systems with bounded actuators, a continuous family of controllers is designed which avoids control saturation for a given worst case disturbance. The control law acts as a scheduling scheme which, at each time, based on the closed-loop behavior (i.e., the closed-loop state vector) out of the continuous family selects a controller that provides the best performance while avoiding saturation. Graphically, the resulting scheduling scheme relies on *slab* regions in phase plane. It is shown that the most aggressive controller can be set to be a nominal controller designed without any regard to the saturation bounds, resulting in an anti-windup scheme. Among advantages of this anti-windup protection scheme are its improved performance while the nominal controller is mildly saturated and its applicability to open-loop unstable plants. The benefits of the proposed technique are demonstrated through two numerical examples.

Key words: Actuator saturation; scheduled controllers; anti-windup; .

1 Introduction

Unavoidable actuator limitation has made controller design for systems with bounded actuators an important area of research for decades. Early attempts led to the development of the two-step *Anti-Windup* schemes, in which first a nominal linear controller is designed for the small signal region; then the nominal controller is augmented with an anti-windup loop to address the undesirable behavior that saturation creates. Recently, in [1] and [2], for example, methodical approaches with rigorous stability and performance guarantees have become available for anti-windup synthesis (see also [3]-[5]).

In recent years, a number of approaches in which the saturation nonlinearity is taken into account, *explicitly*, at the controller design stage have been developed (e.g., see references [6]-[9]). In the explicit approach, resulting (often nonlinear) controllers typically provide performance guarantees that are better than those of the open-loop and are often applicable to open-loop unstable systems, as well. These approaches can be used for a variety of objectives (reducing $\mathcal{L}_2$ or peak-to-peak gains, etc.), though the nominal or small signal controller is often not nearly as desirable as the one used in the anti-windup approach (since the latter is obtained by ignoring saturation constraints with a focus on high performance and small signals). This particular disadvantage becomes more critical if saturation is expected to be infrequent. To alleviate this undesirable characteristic, a family of controllers in some form of scheduling, in response to the closed-loop behavior to disturbances or commands, is often used (e.g., [11]-[13]).

The motivation for this paper is situations where a high performance nominal controller is available but the periods and severity of saturation could be significant and varied, during which guaranteed performance – better than that of the open-loop – is desired. Examples include wind hazard, secondary structural elements in earthquake engineering, or systems with unstable open-loop.

In the first step, we follow [11] to obtain a continuous family of controllers with increasing levels of aggressiveness or performance which will avoid saturation for a given bound on the worst case exogenous input. For simplicity, we use the same technique to obtain the family of controllers as in [11]. The main difference is the way this controller is implemented. Instead of reliance on ellipsoids, which tend to result in significant conservatism, we select the controller based on $x(t)$ (state in

* Parts of this paper was presented at the 2011 IFAC 18th World Congress. Corresponding author S. Sajjadi-Kia. Tel. +1-949-743-4934.

Email addresses: skia@uci.edu (Solmaz Sajjadi-Kia), fjabbari@uci.edu (Faryar Jabbari).

case of state feedback) or $x_c(t)$ (compensator state in case of dynamic output feedback) and seek the most aggressive controller that can be implemented among the family of controllers. Graphically, the resulting scheduling scheme relies on *slab* regions in the phase plane. We also discuss new insights, from establishing performance guarantees or (local) Input-to-State Stability (ISS) to the role played by the parameter that controls how fast the controller can be made more aggressive.

We use a mild assumption: an upper bound to the peak disturbance is known. Given the scheduling used, this bound can be chosen with a great deal of safety margin. Since cases where disturbances are truly unbounded are relatively rare, we consider this assumption mild.

Next, we show the most aggressive controller of the continuous family can be set to be the nominal controller designed separately without any regard to actuator bounds. Then, the rest of the family of controllers ensure stability and performance once the nominal controller is saturated, resulting in an anti-windup scheme with scheduled structure. Scheduling has been attempted in the anti-windup approach before. For example, in [14] scheduling is used to improve the system performance (transients) *after* it re-enters the small signal domain. However, here the proposed approach uses scheduling *during* nominal controller saturation.

Throughout the paper, given a matrix, we use $He\{X\}$ to denote $X + X^T$, whenever space does not allow the full expression. Similarly, in symmetric matrices some of the off diagonal terms are replaced with '$\star$.'

2 Preliminaries and Problem Definition

Consider an open-loop plant with plant state $x \in R^n$, control input $u \in R^{n_u}$ with bounds $|u_i| < u_{max_i},\ i = 1 : n_u$, exogenous external input $w \in R^{n_w}$, and measured and controlled outputs $y \in R^{n_y}$ and $z \in R^{n_z}$:

$$\begin{cases} \dot{x} \ = \ Ax + B_1 w + B_2 u \\ z \ = \ C_1 x + D_{11} w + D_{12} u \ . \\ y \ = \ C_2 x + D_{21} w + D_{22} u \end{cases} \tag{1}$$

The objective here is to design controllers that make the closed-loop system internally stable with a guaranteed disturbance attenuation level without violating saturation bounds. Since the aim is a method that also applies to open-loop unstable systems, such results will be necessarily local. Here, we assume a possibly conservative estimate is known for disturbance $w(t)$, as in

Assumption 1: $w^T(t)w(t) \leq w_{max}^2$, for a known w_{max}.

Stability, acceptable performance, etc. are then guaranteed for disturbances satisfying this bound.

We seek a continuous family of controllers in either full-state feedback controller form, $y = x$:

$$\Sigma_c(\rho) \ \sim \ u = K(\rho)x$$

or dynamic output feedback form

$$\Sigma_c(\rho) \ \sim \ \begin{cases} \dot{x_c} = A_c(\rho)x_c + B_c(\rho)y \\ u(t) = C_c(\rho)x_c \end{cases} \tag{2}$$

where $x_c \in R^{n_c}$ and ρ is the scheduling parameter, in which larger ρ corresponds to higher performance controller. Then, the closed-loop system can be obtained by substituting for u in (1) with $\tilde{x} = x$ in state-feedback case and $\tilde{x} = [x^T \ x_c^T]^T$ in dynamic output feedback case as the closed-loop states. We use the following representation for the closed-loop system:

$$\Sigma_{cl}(\rho) \ \sim \ \begin{cases} \dot{\tilde{x}} \ = \ \tilde{A}(\rho)\tilde{x} + \tilde{B}(\rho)w \\ z \ = \ \tilde{C}(\rho)\tilde{x} + D_{11}w \\ u \ = \ \mathcal{G}(\rho)\tilde{x} \end{cases} \tag{3}$$

where $\mathcal{G}(\rho) = K(\rho)$ in state feedback controller, and $\mathcal{G}(\rho) = [0 \quad C_c(\rho)]$ in output feedback case, and closed-loop matrices $\tilde{A}(\rho)$, etc. are obvious matrices.

The scheduling parameter ρ is chosen such that, among controllers that meet the saturation bound, the highest performance controller is used, that is, at time t, $\rho(t)$ is chosen as the largest value such that $\mathcal{G}(\rho(t))\tilde{x}(t)$ is below the saturation bound (see (9) below). A constraint that comes up naturally is the rate with which this ρ can be increased (d_{max} below). These concepts are introduced formally in the theorem below, with more discussions to follow.

3 Continuous Family of Controllers to Avoid Saturation

Here, we focus on designing a controller which avoids saturation for disturbances satisfying Assumption 1. To reduce the effects a conservative estimation for w_{max}, we use a scheduling scheme that results in guaranteed bounds for disturbances smaller than w_{max}, automatically, as shown below.

Before stating the technical discussion, a clarification on notation might be helpful. Here, we use parameter 'r' as the index to denote the continuous family of Lyapunov functions, ellipsoids, and controllers – for example in the computational stage. Once these are obtained, we use ρ to denote the scheduling parameter that varies, and is calculated, in time and on-line. The controller used at time t is obtained by using $u = \mathcal{G}(\rho(t))\tilde{x} = \mathcal{G}(r)_{|_{r=\rho(t)}}\tilde{x}$.

The basic design methodology is the following. Consider two bounds for the disturbance: $w_{min}^2 \leq w^T(t)w(t) \leq w_{max}^2$. For any r satisfying $w_{min}^2 \leq 1/r \leq w_{max}^2$, we find a controller $\Sigma_c(r)$ that has the following properties: (i) the closed-loop system is internally stable (indeed locally ISS), with a performance (e.g., peak to peak gain) $\delta(r)$ that has the property $\delta(r_2) \leq \delta(r_1)$ if $r_1 < r_2$, and (ii) the ellipsoid

$$\mathcal{E}(P(r), 1/r) = \{\tilde{x} : \tilde{x}^T P(r)\tilde{x} \leq 1/r\} \tag{4}$$

is contained in the linear region of the controller and is the invariant set for the closed-loop as long as $w^T(t)w(t) \leq 1/r \ \ \forall\, t$.

To keep track of the ellipsoids and enforce non-increasing $\delta(r)$, we require that $P(r_2) \geq P(r_1)$ if $r_2 > r_1$, or in other word $dP(r)/dr \geq 0$ (or $dQ(r)/dr \leq 0$ where $Q(r) = P(r)^{-1}$) . This leads to the ellipsoids $\mathcal{E}(P(r), 1/r)$ be strictly nested.

The following Lemma provides the design matrix inequalities (MIs) and the properties of continuous family of controllers.

Lemma 1 *Consider the closed-loop system (3) with control input bounds of $|u_i| < u_{max_i}$, $i = 1 : n_u$ and disturbances satisfying Assumption 1. For a set of constants d_{max}, $\rho_{min} = 1/w_{max}^2$, and ρ_{max}, suppose there exist a $\mathcal{C}^1$ function $Q(r) \in R^{(n+n_c)\times(n+n_c)}$ (for state-feedback $n_c = 0$) with $dQ/dr \leq 0$ and a scalar $\alpha > 0$ that solve the following optimization problem for all $r \in [\rho_{min}, \rho_{max}]$:*

$$\text{minimize } \ J, \ \text{ where } J = \frac{1}{\rho_{max} - \rho_{min}} \int_{\rho_{min}}^{\rho_{max}} \delta(r) dr, \ \ s.t. \tag{5}$$

$$\begin{pmatrix} Q(r) & Q(r)\tilde{C}(r)^T \\ \star & \delta(r) I \end{pmatrix} > 0 \tag{6}$$

$$\begin{pmatrix} He\{\tilde{A}(r)Q(r)\} + \alpha Q(r) - d_{max} \frac{d}{dr} Q(r) & \star \\ \tilde{B}(r)^T Q(r) & -\alpha I \end{pmatrix} < 0 \tag{7}$$

$$\begin{pmatrix} Q(r) & \star \\ [\mathcal{G}(r)]_i\, Q(r) & r u_{max_i}^2 \end{pmatrix} > 0 \quad \forall i = 1 : n_u \tag{8}$$

where $[.]_i$ indicates the i^{th} row of the matrix. Furthermore, consider a function $\rho(t)$ where, at each t, ρ is the largest value that satisfies

$$|\,[\mathcal{G}(\rho)]_i \tilde{x}(t)| \leq u_{max_i} \quad \forall i = 1 : n_u, \quad s.t. \tag{9}$$

$$\rho_{min} \leq \rho(t) \leq \rho_{max}, \quad -\infty \leq \dot{\rho}(t) \leq d_{max} \tag{10}$$

where $\tilde{x}$ is the state of the closed-loop system. Then, for disturbances that, for some $\hat{w}$ as peak bound, meet $w(t)^T w(t) \leq \hat{w}^2 = 1/\hat{\rho} \leq w_{max}^2$, $\forall t \geq 0$, the control law $u = \mathcal{G}(\rho)\tilde{x}$ satisfies the followings for any initial condition $\tilde{x}(0) \in \mathcal{E}(Q(\rho_{max})^{-1}, 1/\hat{\rho})$:

(i) The closed-loop state $\tilde{x}(t)$ remains in $\mathcal{E}(Q(\hat{\rho})^{-1}, 1/\hat{\rho})$.
(ii) The closed-loop system is locally ISS.
(iii) The closed-loop system satisfies

$$z(t)^T z(t) < \delta(\hat{\rho})\, w(t)^T w(t). \tag{11}$$

PROOF. Consider the following Lyapunov function

$$V(\tilde{x}(t), \rho) = \tilde{x}(t)^T Q(\rho)^{-1}\, \tilde{x}(t), \quad Q(\rho) > 0. \tag{12}$$

Part (i): To estimate the invariant set for the closed-loop, we use the standard approach for peak bonded disturbance; i.e., $Q(\rho)^{-1} > 0$ and scalar $\alpha > 0$ such that

$$\dot{V}(\tilde{x}(t), \rho) + \alpha(V(\tilde{x}(t), \rho) - w(t)^T w(t)) < 0. \tag{13}$$

Since the initial conditions are in $\mathcal{E}(Q(\rho_{max})^{-1}, 1/\hat{\rho})$, we start with the initial value of $V(t)$ to be less than $1/\hat{\rho}$. Inequality (8) guarantees that for $w^T(t)w(t) \leq 1/r$, the controller $\mathcal{G}(r)$ satisfies $|u_i| \leq u_{max}$. Then, there exists a controller that does not violate the saturation bound with a ρ that is at least as large as $\hat{\rho}$, due to the control law in (9). Next, note that by substituting for V and $\dot{V}$, we can rewrite (13) in the following form:

$$\begin{pmatrix} He\{\tilde{A}(\rho)Q(\rho)\} + \alpha Q(\rho) - \frac{dr}{dt}\frac{d}{dr} Q(r)|_{r=\rho} & \star \\ \tilde{B}(\rho)^T Q(\rho) & -\alpha I \end{pmatrix} < 0 \tag{14}$$

We have required $\frac{d}{dr} Q(r) \leq 0$ and will enforce (on-line) $\dot{r} \in [-\infty, d_{max}]$. For $\dot{r}(t) = -\infty$, the above inequality is trivial and, as a result, (7) establishes (13). Then, using standard arguments, it can be shown that the closed-loop state vector under the control law in (9-10) remains in $\mathcal{E}(Q(\hat{\rho})^{-1}, 1/\hat{\rho})$ and $V(t) \leq 1/\hat{\rho}$.

Part (ii): Applying the Comparison Lemma ([16]) to (13) for $w(t)^T w(t) \leq \hat{w}^2$, we obtain:

$$V(\tilde{x}(t), \rho(t)) < (V(\tilde{x}(0), \rho(0)) - 1/\hat{\rho}) e^{-\alpha t} + 1/\hat{\rho} \tag{15}$$

As explained in part (i), we have $V(\tilde{x}(0), \rho(0)) < 1/\hat{\rho}$. Then, inequality (15) implies that at any time t, we have $V(\tilde{x}(t), \rho(t)) < 1/\hat{\rho}$. Note that

$$\lambda_{min}(Q(\rho)^{-1})\|\tilde{x}(t)\| \leq V(\tilde{x}(t), \rho(t)) \leq \lambda_{max}(Q(\rho)^{-1})\|\tilde{x}(t)\|$$

Then (15) can be written as

$$\|\tilde{x}(t)\| < \tfrac{1}{\lambda_{min}(Q(\rho)^{-1})} \left((V(\tilde{x}(0), \rho(0)) - 1/\hat{\rho}) e^{-\alpha t} + 1/\hat{\rho}\right)$$

By the standard definition ([16]), this implies local ISS.

Part (iii) is enforced by (8) and the result of standard approach of deriving bounds on $\|z(t)\|$ using invariant ellipsoids. Due to $dQ/dr < 0$, larger r relaxes (6), thus we have $\delta(r_1) \leq \delta(r_2)$ if $r_1 > r_2$, which is the motivation for looking for the largest ρ that satisfies (9).

While the performance measure used here is the peak-to-peak gain estimate (upper bound), several other variations can be used with rather standard modification, e.g., energy bounded disturbance, $\mathcal{L}_2$, or energy-to-peak performance measures. An alternative (convex) formulation for (6)-(8) are presented in (17)-(19) below.

Remark 1 *In many cases w_{max} can be quite conservative, since it is an estimate of the worst peak disturbance. The proposed scheduling address this problem since if the worst peak is $\hat{w} < w_{max}$, the controller automatically guarantees better performance, so that less severe operating conditions automatically result in higher levels of performance. Note that this $\hat{w}$ is not needed to be known – only w_{max} is used in the design step.*

Remark 2 *The parameter d_{max} plays an important role since it controls the 'shape' of the underlying Lyapunov* *matrices. As d_{max} is increased, it would allow more rapid increases in ρ and thus more rapid increases in the value of the Lyapunov function V. Note that an infinitely large d_{max} forces the Lyapunov matrices to become constant – thus leading to possibly quite conservative results.*

We refer to this implementation as the 'slab' approach, since the linear regions of any controller is a slab shape in the hyperplane. The controller searches directly among control gains to find the one associated with largest ρ, as opposed to the approach in [11] which was 'ellipsoid' based and searched for the smallest ellipsoid containing the state vector with the associated controller avoiding the saturation bound. The larger zone of action for slab based implementation can result in better performance by invoking the full potential of the controller. As the numerical example below shows, the ellipsoid based approach can have significant conservatism.

3.1 Convex LMIs to Design Full-order Dynamic Output Feedback Control

Expanding the optimization problem in Lemma 1 to a convex form for state-feedback is quite straightforward: We substitute the closed-loop matrices in the MIs and replace the non-convex term $K(r)Q(r)$ with the intermediate variable $F(r)$. Once $Q(r)$ and $F(r)$ are obtained the controller is recovered by $K(r) = F(r)Q(r)^{-1}$. To convexify the dynamic feedback case, we follow the technique used in [17] or [11] where full order (i.e., $n_c = n$) controllers are obtained by using the following structure for the Lyapunov matrix, without losing any generality (see [10], page 716) using $S(r) = X(r) - Y^{-1}$:

$$Q(r)^{-1} = \begin{pmatrix} Y & -Y \\ -Y & S(r)^{-1} + Y \end{pmatrix}, \quad Q(r) = \begin{pmatrix} X(r) & S(r) \\ S(r) & S(r) \end{pmatrix} \quad (16)$$

To eliminate complications in calculation and implementation of $A_c(r)$ (e.g., avoiding $\dot{\rho}$ in A_c), similar to many quasi linear parameter varying techniques, we use a constant Y. Using this structure and applying some standard congruent transformations, the optimization problem in Lemma 1 can be restated by replacing (6)-(8) with the following convex forms, respectively (with a line search on α) -see [17] for details. The nestedness condition, $dQ(r)/dr \leq 0$, here becomes $dX(r)/dr \leq 0$.

$$\begin{pmatrix} X(r) & \star & \star \\ I & Y & \star \\ C_1 X(r) + D_{12} F(r) & C_1 & \delta(r) I \end{pmatrix} > 0 \quad (17)$$

$$\begin{pmatrix} \Omega_{11}(r) - d_{max} \frac{d}{dr} X(r) & \star & \star \\ A^T + L(r) + \alpha I & \Omega_{22}(r) + \alpha Y & \star \\ B_1^T & B_1^T Y + D_{21}^T G(r)^T & -\alpha I \end{pmatrix} < 0 \quad (18)$$

where $\Omega_{11}(r) = He\{AX(r) + B_2 F(r)\} + \alpha X(r)$ and $\Omega_{22}(r) = He\{YA + G(r)C_2\}$,

$$\begin{pmatrix} X(r) & I & [F(r)]_i^T \\ I & Y & 0 \\ [F(r)]_i & 0 & r u_{max_i}^2 \end{pmatrix} > 0 \quad \forall i = 1 : n_u \quad (19)$$

for $r \in [\rho_{min}, \rho_{max}]$. Then, the controller is defined by $C_c(r) = F(r)S(r)^{-1}$, $A_c(r) = (A - B_c(r)C_2)X(r)S(r)^{-1} + B_2 C_c(r) - Y^{-1} L(r) S(r)^{-1}$, and $B_c(r) = -Y^{-1} G(r)$.

Remark 3 *Here, we show that $B_c(r)$ can be set to a constant B_c, without loss of generality. This plays a key role in extending the results to anti-windup scheme in Section 4. Consider the invariant set MI (18), which is the key inequality containing variables Y, B_c and G. Applying Elimination Lemma (see Section 2.6.2 of [15]) and eliminating $L(r)$, (18) is equivalent to*

$$\begin{pmatrix} M(r) + \alpha X - \frac{dX}{dr} d_{max} & \star \\ B_1^T & -\alpha I \end{pmatrix} < 0 \quad (20)$$

$$\begin{pmatrix} He\{YA + G(r)C_2\} + \alpha Y & \star \\ B_1^T Y + D_{21}^T G(r)^T & -\alpha I \end{pmatrix} < 0 \quad (21)$$

Now, $G(r)$ appears only in (21) and there it is the only variable that depends on r. Therefore, without any loss of generality, one can replace $G(r)$ with constant G for all values of r, which is equivalent to constant B_c, since we use a constant Y.

The optimization problem in Lemma 1 can be solved, numerically, through different discretization techniques available in the literature (e.g., [13] and [17]). In the Appendix, we present a technique based on the linear splines of [17] (also used in [11]).

4 Anti-windup via Overriding Controllers

In addition to reduced conservatism, a key benefit of the slab-based scheduling is the possibility of using it in a fashion similar to the anti-windup augmentation, even for unstable systems. Here, we can use the nominal controller as the most aggressive controller, corresponding to ρ_{max}, hence establishing an anti-windup scheme. However, instead of an augmentation loops often seen in the traditional anti-windup, this technique replaces the nominal controller with a family of controllers via scheduling once the nominal controller is saturated.

Suppose $H(s)$ is a high performance pre-designed nominal controller for (1) with a full order (i.e., $n_c = n$) minimal state space realization:

$$\Sigma_{c,nom} \sim \begin{cases} \dot{x}_{c,nom} = A_{c,nom} x_c + B_{c,nom} y \\ u = C_{c,nom} y \end{cases} \quad (22)$$

The closed-loop system under this controller saturates for some disturbance satisfying Assumption 1. The idea is to set $\Sigma_c(\rho_{max}) = \Sigma_{c,nom}$ and use the continuous family of the overriding controllers with the corresponding close-loop satisfying conditions of Lemma 1 to guarantee stability and establish performance.

To obtain the controllers, we start with the design MIs of Section 3.1. For $r = \rho_{max}$ we use the nominal controller and thus there is no need to search for controller gain at this r. As a result, for $r=\rho_{max}$, the intermediate variables $L(r)$, $G(r)$ and $F(r)$ are not necessary. However, substituting the nominal controller matrices in $L(r)$ and $F(r)$, as shown below, makes the design MIs non-convex (recall $S(r)=X(r)-Y^{-1}$):

$$L(r=\rho_{max}) = YB_2C_{c,nom}S(\rho_{max}) - YA_{c,nom}S(\rho_{max}) + Y(A-B_{c,nom}C_2)X(\rho_{max})$$

$$F(r=\rho_{max}) = C_{c,nom}S(\rho_{max})$$

To convexify the design MIs (17)-(19) for the anti-windup problem, we use the following steps. First, as we discussed in Remark 3, we search for a single B_c. Therefore, we have $B_c = B_{c,nom}$, for all r. As a result, variable G should be replaced with $-B_{c,nom}Y$. Then, we replace $L(r)$ with $Y(A-B_{c,nom}C_2)X(r)+YB_2(_c(r)S(r)-YA_c(r)S(r)$, and $F(r)$ with $C_c(r)S(r)$ and replace $S(r)$ with $X(r)-Y^{-1}$. Then to convexify, we preform a congruent transformation with transfer matrix $Diag[I, V, I]$, where $V=Y^{-1}$ on (17)-(19). Lastly, for $r<\rho_{max}$, we define $\tilde{L}(r)=(A-B_{c,nom}C_2)X(r)+B_2C_c(r)S(r)-A_c(r)S(r)$ and $F(r)=C_c(r)S(r)$ as the new search variables. The result is summarized in the following lemma.

Lemma 2 *Consider the closed-loop system (3) with control input bounds of $|u_i|<u_{max_i}$, $i=1:n_u$ and disturbances satisfying Assumption 1 and a pre-deigned nominal controller (22) associated with $\rho=\rho_{max}$. For a set of constants d_{max}, $\rho_{min}=1/w^2_{max}$, and ρ_{max}, suppose there exist a $\mathcal{C}^1$ function $X(r), \tilde{L}(r)\in R^{2n\times 2n}$ with $dX/dr\le 0$, $F(r)\in R^{n_u\times n}$, positive definite matrix $V\in R^{n\times n}$ and a scalar $\alpha>0$ that solve the following optimization problem for all $r\in[\rho_{min},\rho_{max}]$:*

$$\text{minimize } J, \text{ where } J=\frac{1}{\rho_{max}-\rho_{min}}\int_{\rho_{min}}^{\rho_{max}}\delta(r)dr, \quad s.t.(23)$$

<u>for $\rho_{min}\le r<\rho_{max}$</u>

$$\begin{pmatrix} X(r) & \star & \star \\ V & \mathbf{V} & \star \\ C_1X(r)+D_{12}F(r) & C_1V & \delta(r)I \end{pmatrix} > 0 \tag{24}$$

$$\begin{pmatrix} \Phi_{11}(r)-d_{max}\frac{d}{dr}X(r) & \star & \star \\ VA^T+\tilde{L}(r)+\alpha V & \Phi_{22}+\alpha V & \star \\ B_1^T & B_1^T-D_{21}^TB_{c,nom}^T & -\alpha I \end{pmatrix} < 0 \tag{25}$$

$$\begin{pmatrix} X(r) & \star & \star \\ V & V & \star \\ F(r) & 0 & ru^2_{max}I \end{pmatrix} > 0 \tag{26}$$

where $\Phi_{11}(r)=He\{AX(r)+B_2F(r)\}+\alpha X(r)$ and $\Phi_{22}(r)=He\{AV-B_{c,nom}C_2V\}$

<u>for $r=\rho_{max}$</u>

$$\begin{pmatrix} X(r) & \star & \star \\ V & V & VC_1^T \\ C_1X(r)+D_{12}C_{c,nom}(X(r)-V) & \star & \delta(r)I \end{pmatrix} > 0 \tag{27}$$

$$\begin{pmatrix} \Psi_{11}(r)-d_{max}\frac{d}{dr}X(r) & \star & \star \\ \Psi_{21}(r)+\alpha V & \Psi_{11}(r)+\alpha V & \star \\ B_1^T & B_1^T-D_{21}^TB_{c,nom}^T & -\alpha I \end{pmatrix} < 0 \tag{28}$$

$$\begin{pmatrix} X(r) & \star & \star \\ V & V & 0 \\ C_{c,nom}(X(r)-V) & 0 & \rho_{max}u^2_{max}I \end{pmatrix} > 0. \tag{29}$$

where $\Psi_{11}(r)=He\{(A-B_2C_{c,nom})X(r)-B_2C_{c,nom}V\}+\alpha X(r)$, $\Psi_{21}(r)=VA^T+\alpha V+(A-B_{c,nom}C_2)X(r)+B_2C_{c,nom}(X(r)-V)-A_{c,nom}(X(r)-V)$ and $\Psi_{22}(r)=He\{AV-B_{c,nom}C_2V\}$.

Once V, $F(r)$, $X(r)$, and $\tilde{L}(r)$ are obtained the controller matrices at $r<\rho_{max}$ can be obtained from $A_c(r)=(A-B_{c,nom}C_2)X(r)S(r)^{-1}+B_2C_c(r)-\tilde{L}(r)S(r)$ and $C_c(r)=F(r)S(r)^{-1}$, where $S(r)=X(r)-V$.

In case of full state feedback problem with pre-deigned controller gain of K_{nom}, the design procedure is much simpler. Here, we directly use the MI's of Lemma 1. For $r=\rho_{max}$, We use the gain K_{nom} and for the rest of the family we replace $K(r)Q(r)$ with $F(r)$. The resulting set is convex (modulo α) under the variables $Q(r)$ and $F(r)$. Once the problem is solved the controller gains for $r\in[r_{min},r_{max})$ can be recovered from $K(r)=F(r)Q(r)^{-1}$.

For computing the controller, the spline method in the Appendix seems well suited, since the constraint of the fixed nominal controller affects only the search at that particular r. The discretized algorithm for (24)-(29) is presented in the Appendix.

While not a requirement, in our experience, for a given nominal controller say with transfer function $H(s)$, a state space representation compatible with a Lyapunov matrix of the form (16) helps improve the guaranteed performance levels obtained. The following is one possible approach: We start with any state space representation and seek the Lyapunov function that minimizes δ subject to the discretized version of (6)-(8) without the derivative terms. Then through a standard congruent transformation (see, e.g., [10], page 716), the state space representation of the nominal controller is transformed so that the Lyapunov matrix has the structure in (16) and the MIs for ρ_{max} are satisfied.

Remark 4 *In traditional anti-windup the difference between the input and output of the saturation element is the signal to activate the anti-windup. Thus, in static anti-windup, when the nominal controller leaves the saturation zone, the nominal closed-loop system is recovered immediately. In dynamic anti-windup compensation (see,*

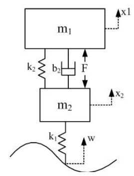

Fig. 1. Active suspension model

e.g., [2]), some residual dynamics can last for a while but since the activating signal is zero, it tends to die off. In the scheduling used here, the parameter d_{max} regulates how fast we can return to the nominal controller. Unless it is set to infinity, the return to the nominal controller will not be immediate. Recall that as d_{max} gets larger, the Lyapunov matrix Q (technically its inverse) tends to a constant matrix (recall (7)), often yielding a far more conservative family of controllers, i.e., much higher $\delta(r)$. As a result, in many cases, practically for open-loop unstable system, a solution might not be found.

4.1 Anti-windup for Nominal Controllers with an Order less than the Plant Order

In the foregoing development, we assumed that the nominal controller has the same order as the plant. We now relax this condition to the controller having an order equal or less than that of the plant. Consider the system with a dynamic output feedback nominal controller (22) of order $n_c < n$. We add the fictitious states $\hat{x} \in R^{n-n_c}$ with some stable dynamics, i.e., $\hat{A}_c$ has negative eigenvalues, to the nominal controller as follows:

$$\begin{cases} \begin{bmatrix} \dot{x}_c \\ \dot{\hat{x}}_c \end{bmatrix} = \begin{bmatrix} A_{c,nom} & 0 \\ 0 & \hat{A}_c \end{bmatrix} \begin{bmatrix} x_c \\ \hat{x}_c \end{bmatrix} + \begin{bmatrix} B_{c,nom} \\ 0 \end{bmatrix} y \\ u = \begin{bmatrix} C_{c,nom} & 0 \end{bmatrix} \begin{bmatrix} x_c \\ \hat{x}_c \end{bmatrix} \end{cases} \tag{30}$$

This modification does not alter the nominal controller commands, it only changes the state space order of the nominal controller and makes it full order. Thus, one can use the control algorithm developed above for systems with nominal controllers with orders $n_c < n$.

5 Numerical Example

5.1 Example 1: Slab versus Ellipsoidal Scheduling

The numerical example here, taken from [11], shows the improvement due to the slab conditions, as opposed to using ellipsoids of [11]. To be consistent with [11], we implement the $\mathcal{L}_2$-gain version of Lemma 1. The MIs can be obtained from [11], only the algorithm to select the controller - $\rho(t)$ in (40) of the Appendix – is different. Consider an active suspension control system model of

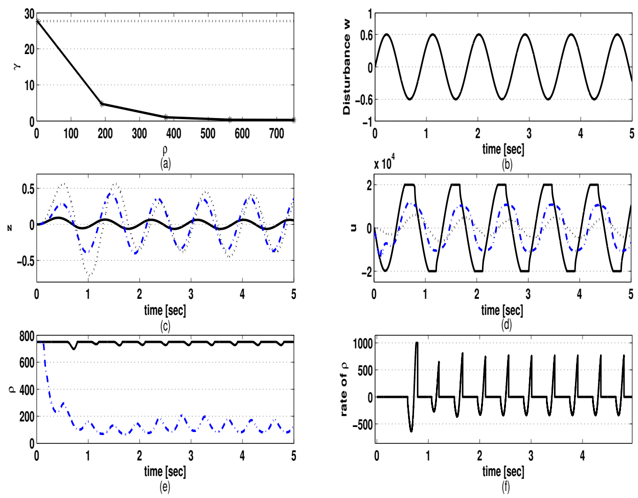

Fig. 2. Simulation results; solid line:slab condition, dash-dotted: ellipsoidal condition, dotted: nonscheduled

automobile shown in Fig. 1. The disturbance, w is the vertical irregularities along the road. The control input F is applied between the body of automobile, mass m_1, and the tires, mass m_2. The numerical values we use are: $m_1 = 1500\ kg,\ m_2 = 60\ kg,\ k_1 = 190 \times 10^3 N/m,\ k_2 = 35 \times 10^3\ N/m,\ b_2 = 1000\ Ns/m$. We assume that the vertical position and the velocity of the automobile (x_1 and x_3) are available for feedback and $z = x_1$ as the controller output. We design a scheduled controller against disturbances up to 60 cm and and $u_{max} = 2 \times 10^4\ N$. The disturbance values here are larger and the saturation limit lower than those in [11] since using the values of [11] resulted in a slab based controller that would saturate infrequently or not face the d_{max} limit. The more demanding conditions show the benefits of the new approach more clearly, though it can result in somewhat different solutions for the design variables (e.g., α).

Here, we use $c_k = 1$. We also compare the simulation results with those obtained for 'nonscheduled' controllers, i.e., controller obtained with $\rho_{max} = \rho_{min}$. We design our scheduled controller with the following parameters: $d_{max} = 1000$, $\alpha = 0.78$, $n_r = 5$, and $\rho_{max} = 750$. Figure 2.(a) shows the $\mathcal{L}_2$ gain, γ vs. ρ. As expected, in the case of scheduled controller, as ρ becomes larger, i.e. the states are closer to the small signal region, we have more aggressive controllers.

Figure 2.(c)-(f) depicts the simulation results for disturbance of Fig. 2.(b). Figure 2.(c) shows both scheduled schemes have better performance than the nonscheduled controller. Figure 2.(e) shows the time history of scheduling parameter ρ. Although the peak value of the disturbance is w_{max}, the controller scheme has been able to apply more aggressive controller when it was possible. Figure 2.(c) also shows that the slab scheme renders a better performance compared to the ellipsoidal scheme. This can also be explained by inspection of Fig. 2.(e), where slab condition uses more aggressive con-

troller, larger ρ, compared to the ellipsoidal scheduling scheme. Time history of the control input, Fig. 2.(d), shows in the ellipsoidal algorithm switches happen (i.e, a lower gain controller is used) even though the control input is well below the saturation bound, a typical conservatism faced in techniques that rely on ellipsoids. Higher utility of the control capacity in the slab based controller leads to significant improvement.

5.2 *Example 2: Anti-windup*

Consider the following unstable plant:

$$\dot{x} = \begin{bmatrix} -0.05 & 1 \\ 0.1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w$$
$$y = x_1, \quad z = x_2$$

Here the nominal controller is a H_∞ full order dynamic output feedback designed without any regard for saturation bound of $u_{max} = 10$. The following realization corresponds to the Lyapunov matrix of the form in (16).

$$\left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right]_{nom} = \left[\begin{array}{cc|c} -13.5879 & -12.8475 & 13.5060 \\ -50.8441 & -6.8648 & 50.8774 \\ \hline -0.0304 & -14.2409 & 0 \end{array}\right].$$

For disturbances with peak value around 2 the nominal controller saturates. The goal here is to extend the tolerable disturbance bound to $w_{max} = 15$. We set $\Sigma_{c,nom}$ as the $\Sigma_c(\rho_{max})$ in the scheduled scheme, and use the following parameters: $n_r = 10$, $d_{max} = 5$, and $\rho_{max} = \frac{1}{1.1^2}$ (equivalent to $w_{min} = 1.1$). Note that w_{min} is a value picked by the designer and must be set to a value smaller than the maximum guaranteed disturbance level for the nominal controller. Figure 3.(a) shows the performance variation with ρ: as ρ decreases system sacrifices the performance to decrease the control gain and the possibility of saturation.

Figure 3.(c)-(f) show the results of the simulation for the disturbance with peak bound of 15, shown in Fig. 3.(b). Figure 3.(c) indicates that once saturated, the nominal closed loop is unstable. Implementing the schedule controller stabilizes the system and eventually recovers the nominal controller. The scheduling parameter ρ in Fig. 3.(e) and its rate in Fig. 3.(f) show that once the nominal controller saturates, the control law immediately jumps to the controller with lowest gain As discussed earlier, because of the limit on d_{max} however, the system return can be gradual. Figure 3.(d) depicts the time history of the control input.

6 Appendix

To reduce the optimization problem in Lemma 2 to a finite set of MIs, we invoke the results of [17] on using linear spline approximations for continuous MIs.

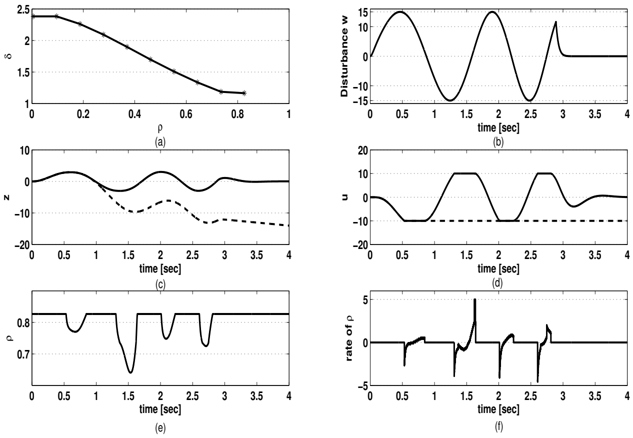

Fig. 3. Simulation Results: dashed lines are for the nominal system with no anti-windup, and the solid lines are for the system with anti-windup

To estimate the functions, we partition the range $r \in [\rho_{min}, \rho_{max}]$ into:

$$\frac{1}{w_{max}^2} = \rho_{min} = r_1 < r_2 < \cdots < r_{n_r} = \rho_{max} = \frac{1}{w_{min}^2} \quad (31)$$

Then, any matrix variable $H(r)$ with $r \in [r_k, r_{k+1}]$ is obtained with a linear spline:

$$H(r) = H_k + \frac{r - r_k}{r_{k+1} - r_k}(H_{k+1} - H_k). \quad (32)$$

The search then is for $H_k\,'s$. The technical details follow the treatment used in [11]. For example, $\mathcal{C}^1$ continuity of the Lyapunov function and of ρ are addressed by small 'smoothing' tricks that exploit the fact that numerical solutions are obtained through strict inequalities. These fixes have no effect on the implementation. They are used to establish equivalency between the spline approximation and the continuous form. Also, details involving the selected number of 'nodes' or computational issues are similar to [17] and [11], and thus are not repeated. It suffices to say that the number of nodes reflects the computational load for off-line calculations but does not alter the on-line effort significantly. In the following due to the limited space, we only give the spline approximation of anti-windup deign of Section 4.

$$\text{minimize} \quad \sum_k c_k \gamma_k \quad (\text{where } c_k > 0 \text{ used as weights}), \text{ s.t.}$$

$$\underline{\text{for } k = 1, \cdots, n_r - 1}$$

$$X_{k+1} < X_k \quad (33)$$

$$\begin{pmatrix} X_k & \star & \star \\ V & V & \star \\ C_1 X_k + D_{12} F_k & C_1 V & \delta_k I \end{pmatrix} > 0 \quad (34)$$

$$\begin{pmatrix} \Phi_{11_k} - d_{max}\Delta X_m & \star & \star \\ V A^T + \tilde{L}_k + \alpha V & \Phi_{22} + \alpha V & \star \\ B_1^T & B_1^T - D_{21}^T B_{c,nom}^T & -\alpha I \end{pmatrix} < 0$$
$$m = (k, k-1) \quad (35)$$

$$\begin{pmatrix} X_k & \star & \star \\ V & V & \star \\ F_k & 0 & r_k u_{max}^2 I \end{pmatrix} > 0 \qquad (36)$$

where $\Phi_{11_k} = He\{AX_k + B_2F_k\} + \alpha X_k$ and $\Phi_{22} = He\{AV - B_{c,nom}C_2V\}$

and for $k = n_r$

$$\begin{pmatrix} X_{n_r} & \star & \star \\ V & V & \star \\ C_1X_{n_r} + D_{12}C_{c,nom}(X_{n_r} - V) & C_1V & \delta_{n_r} I \end{pmatrix} > 0 \qquad (37)$$

$$\begin{pmatrix} \Psi_{11_k} - d_{max}\Delta X_m & \star & \star \\ \Psi_{21} + \alpha V & \Psi_{22} + \alpha V & \star \\ B_1^T & B_1^T - D_{21}^T B_{c,nom}^T & -\alpha I \end{pmatrix} < 0$$

$$m = (n_r, n_r - 1) \qquad (38)$$

$$\begin{pmatrix} X_{n_r} & \star & \star \\ V & V & 0 \\ C_{c,nom}(X_{n_r} - V) & 0 & r_{n_r} u_{max}^2 I \end{pmatrix} > 0 \qquad (39)$$

where $\Psi_{11_k} = He\{(A - B_2C_{c,nom})X_k - B_2C_{c,nom}V\} + \alpha X_k$, $\Psi_{21} = VA^T + \alpha V + (A - B_{c,nom}C_2)X_k + B_2C_{c,nom}(X_k - V)$ $-A_{c,nom}(X_k - V)\Psi_{22} = He\{AV - B_{c,nom}C_2V\}$, and $\Delta X_m = (X_{m+1} - X_m)/(r_{m+1} - r_m)$

We set $\Delta X_0 = \Delta X_{n_r} = 0$. Once V, and F_k, X_k, and $\tilde{L}_k$ are obtained the unknown controller matrices at nodes $k = 1, \cdots, n_r - 1$ can be obtained from $(C_c)_k = F_k S_k^{-1}$ and $(A_c)_k = (A - B_{c,nom}C_2)X_k S_k^{-1} + B_2(C_c)_k - \tilde{L}_k S_k^{-1}$, where $S_k = X_k - V$.

Weights c_k place emphasis on a given range of ρ. For example, if it is expected that saturation violations be mild (i.e., command exceeding capacity modestly), more emphasis should be placed on values close to ρ_{max} while for cases where the worst case disturbance is expected to lead to extended severe saturation violation, more weight near ρ_{min} would be appropriate.

Once the controller gains are obtained at node points, the continuous controller gain is obtained with the spline form in (32). We use the following algorithm to obtain $\rho(t)$ in (9) and eventually the controller gains at any time. Given x_c, for $\forall i = 1 : n_u$ determine

$$k = max\{j = 1, \cdots, n_r : \ |\, [C_c(r_j)]_i x_c(t) \,| \leq u_{max_i}\}$$

If $k = n_r$, then set $\rho(t) = r_{n_r}$ otherwise let $\rho(t)$ be equal to the largest $r \in [r_k, r_{k+1}]$ such that

$$|\, [\mathcal{G}(r)]_i \tilde{x}(t) \,| = |\, [C_c(r)]_i x_c(t) \,| \leq u_{max_i} \qquad (40)$$

where $C_c(r) = F(r)S(r)^{-1}$ and, $S(r)$ and $F(r)$ are spline functions of the form (32).

This procedure finds the maximum value of the ρ that does not lead to violation of the actuator bounds. As discussed earlier, ρ is allowed to decrease as fast as needed but, depending on d_{max}, but it may not increase as fast as (40) might allow. During such periods, the implemented value of ρ is increased with a constant rate of d_{max}, until it matches the value obtained from (40). All of the properties established in Lemma 1 still hold.

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