Dynamic Average Consensus in the Presence of Communication Delay Over Directed Graph Topologies

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Abstract— **This paper analyzes the stability and convergence of a distributed algorithmic solution to dynamic average consensus over strongly connected and weight-balanced digraphs in the presence of communication delay. Our starting point is a distributed coordination strategy that, under continuous-time no-delay communication, achieves practical asymptotic tracking of the dynamic average of the agents' reference signals. For this algorithm, we characterize the admissible communication delay range and study the effect of communication delay on the tracking error bounds. For a given delay value, we use Lambert W function to obtain an estimate on the algorithm's worst rate of convergence. Also, for the strongly connected and weight-balanced digraphs, we establish a relationship between the maximum degree of the network topology and a bound on the size of the delay that the average consensus algorithm can tolerate. Simulations illustrate the results.**

I. Introduction

In the dynamic average consensus problem the objective is to design a distributed algorithm which enables the agents of a network each endowed with a dynamic reference input signal to asymptotically track the dynamic average of these reference inputs. The solution to this problem is of interest in numerous applications such as sensor fusion [1], [2], multi-robot coordination [3], distributed estimation [4], distributed tracking [5] and distributed optimization algorithm design [6], [7]. Our aim in this paper is to study the robustness of a continuous-time dynamic average consensus algorithm to a fixed communication delay.

Literature review: Distributed solutions to dynamic and static consensus problems over network systems have attracted increasing attention in the last decade. Static average consensus, in which the reference signal at each agent is a constant static value, has been studied extensively in the literature(see e.g., [8], [9], [10]). Many aspects of static average consensus problem including analyzing the convergence of the proposed algorithms in the presence of communication delays are studied in the literature (see e.g., [8], [11], [12], [13]). Also, static consensus problem for multi-agent systems with second order dynamics in the presence of communication delay has been studied in [14], [15], [16]. Dynamic average consensus problem has been studied in the literature as well, however the solutions to this problem only guarantee convergence to some neighborhood of the network's dynamic average reference signal (see. e.g., [1], [17], [18], [19], [20], [21] for continuous-time algorithms and [22], [20] for discrete-time algorithms). Although some of these references address important practical considerations such as dynamic average consensus over switching topologies and over networks with event-triggered communication strategy, to the knowledge of the authors there is no study on dynamic average consensus algorithms in the presence of communication time delay. Since delays are inevitable in real systems, it is necessary and beneficial to study the effect of them on dynamic consensus algorithms. This paper intends to fill this gap. The dynamic average consensus algorithm that we consider in this paper is a delay differential equation (DDE). The characterization of solutions of DDEs and their convergence analysis can be found, for example, in [23], [24].

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Statement of contributions: In this paper, we study dynamic average consensus problem in the presence of fixed time delay on the exchanged information between agents over a weight-balanced and strongly connected digraph. Given a time-varying signal for each agent, this problem includes designing a distributed algorithm that allows agents to track the time-varying average of the signals using only the information from neighbors. It is assumed that each agent has access to its own data with no delay, but can only receive delayed information from its neighbors. The solution we consider is the delay free algorithm proposed in [20] for strongly connected and weight-balanced digraphs but in the presence of fixed time delay in communication channels. We obtain an estimate of the admissible range for the communication delay and establish its relation to the degree of the communication network when it is strongly connected and weight-balanced. We also use the Lambert W function to obtain an estimate on the worst rate of convergence of the algorithm. This bound is a function of the given time delay and network topology. Our convergence analysis results in establishing a practical bound on the tracking error. Our analysis shows that when the reference signals are static for delays in admissible bound the algorithm achieves perfect tracking in the presence of the time-delay in communication channels.

Organization: Section II introduces our basic notation, and presents our graph-theoretic notions and terminology. Section III presents the network model and the dynamic average consensus problem in the presence of communication delay. Section IV contains our analysis and study of an algorithmic solution for dynamic consensus problem in the presence of the time delay. Section V presents simulations and Section VI gathers our conclusions and ideas for future work. For convenience of the reader we provide a short review of relevant material from literature on DDE systems in the Appendix. Due to the space limitations, some of the proofs are omitted and will appear elsewhere.

II. Preliminaries

In this section, we introduce our notations and define the graph-theoretic notations and terminologies used throughout the paper.

Notation: we let $\mathbb{R}$, $\mathbb{R}_{>0}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Z}$, and $\mathbb{C}$ denote the set of real, positive real, non-negative real, integer, and complex numbers, respectively. For $s \in \mathbb{C}$, $\mathrm{Re}(s)$, $\mathrm{Im}(s)$ represents its real and imaginary parts, respectively. Moreover, $|s|$ represents its magnitude, i.e., $|s| = \sqrt{\mathrm{Re}(s)^2 + \mathrm{Im}(s)^2}$. Consequently, when $s \in \mathbb{R}$, $|s|$ is its absolute value. For $\mathbf{s} \in \mathbb{R}^d$, $\|\mathbf{s}\| = \sqrt{\mathbf{s}^\top \mathbf{s}}$ denotes the standard Euclidean norm. The transpose of a matrix $\mathbf{A}$ is $\mathbf{A}^\top$. We let $\mathbf{1}_n$ (resp. $\mathbf{0}_n$) denote the vector of n ones (resp. n zeros), and denote by $\mathbf{I}_n$ the $n \times n$ identity matrix. We let $\mathbf{\Pi}_n = \mathbf{I}_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^\top$. When clear from the context, we do not specify the matrix dimensions. For a dynamic signal $\mathbf{u}$, we denote by $\|\mathbf{u}\|_{\text{ess}}$, the (essential) supremum norm , i.e., $\|\mathbf{u}\|_{\text{ess}} = \sup\{\|\mathbf{u}(t)\|,\ t \geq 0\} < \infty$. In a networked system, we distinguish the local variables at each agent by a superscript, e.g., p^i is the local variable of agent $i \in \{1, \dots, N\}$. We represent the aggregate vector of local variables p^i, $i \in \{1, \dots, N\}$, by $\mathbf{p} = (p^1, \dots, p^N)^\top \in \mathbb{R}^N$. We define $\mathfrak{r} = \frac{1}{N}\mathbf{1}_N$ and $\mathfrak{R} \in \mathbb{R}^{(N-1)\times(N-1)}$ such that $[\mathfrak{r} \quad \mathfrak{R}] \in \mathbb{R}^{N \times N}$ is an orthonormal matrix. Notice that $\mathfrak{R}\mathfrak{R}^\top = \mathbf{\Pi}_N$. Therefore, for any $\mathbf{y} \in \mathbb{R}^N$, we can write

$$\|\mathfrak{R}^\top \mathbf{y}\| = \sqrt{\mathbf{y}^\top \mathfrak{R}\mathfrak{R}^\top \mathbf{y}} = \sqrt{\mathbf{y}^\top \mathbf{\Pi} \mathbf{y}} = \sqrt{\mathbf{y}^\top \mathbf{\Pi}\,\mathbf{\Pi} \mathbf{y}} = \|\mathbf{\Pi}\mathbf{y}\|. \tag{1}$$

The Lambert $\mathbf{W}$ function is a set of functions corresponding to the branches of the inverse of $f(s) = s\,\mathrm{e}^s$ where $s \in \mathbb{C}$ and e^s in the exponential function. The Lambert W function is complex and multivalued with infinite number of branches since the function $f(.)$ is not injective. In particular, the equation $s = W(s)\,\mathrm{e}^{W(s)}$ has infinitely many solutions on the complex plane. These solutions are represented by $W_k(s)$ with the branch index k ranging over $\mathbb{Z}$. For all real $s \in \mathbb{R}_{\geq 0}$, the equation $s = W(s)\mathrm{e}^{W(s)}$ has exactly one real solution. It is represented by $W_0(s)$. In particular, for $s \in \mathbb{C}$, we have

$$W_0(s) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} s^n. \tag{2}$$

We use the following result in our developments below.

Lemma 2.1 (Maximum real part of Lambert W function [25]): *For any* $s \in \mathbb{C}$*,*

$$\max\{\mathrm{Re}(W_k(s)) | k \in \mathbb{Z}\} = \mathrm{Re}(W_0(s))$$

is satisfied.

For further discussions on Lambert W function see [25], [26].

Graph theory: in the following, we review some basic concepts from algebraic graph theory following [27]. A *directed graph*, or simply a *digraph*, is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \dots, N\}$ is the *node set* and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the *edge set*. An edge from i to j, denoted by (i, j), means that agent j can send information to agent i. For an edge $(i, j) \in \mathcal{E}$, i is called an *in-neighbor* of j and j is called an *out-neighbor* of i. We denote the set of out-neighbors of an agent $i \in \{1, \dots, N\}$ by $\mathcal{N}^i$. A graph is *undirected* if $(i, j) \in \mathcal{E}$ anytime $(j, i) \in \mathcal{E}$. A *directed path* is a sequence of nodes connected by edges. A digraph is called *strongly connected* if for every pair of vertices there is a directed path connecting them.

A *weighted digraph* is a triplet $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathsf{A})$, where $(\mathcal{V}, \mathcal{E})$ is a digraph and $\mathsf{A} \in \mathbb{R}^{N \times N}$ is a weighted *adjacency* matrix with the property that $\mathsf{a}_{ij} > 0$ if $(i, j) \in \mathcal{E}$ and $\mathsf{a}_{ij} = 0$, otherwise. A weighted digraph is *undirected* if $\mathsf{a}_{ij} = \mathsf{a}_{ji}$ for all $i, j \in \mathcal{V}$. We refer to a strongly connected and undirected graph as a *connected graph*. The *weighted out-degree* and *weighted in-degree* of a node i, are respectively, $\mathsf{d}^i_{\text{in}} = \sum_{j=1}^{N} \mathsf{a}_{ji}$ and $\mathsf{d}^i_{\text{out}} = \sum_{j=1}^{N} \mathsf{a}_{ij}$. A digraph is *weight-balanced* if at each node $i \in \mathcal{V}$, the weighted out-degree and weighted in-degree coincide (although they might be different across different nodes). For weight-balanced digraphs, we define the maximum degree of the digraph as $\mathsf{d}^{\max} = \max\{\mathsf{d}^1_{\text{out}}, \cdots, \mathsf{d}^N_{\text{out}}\} = \max\{\mathsf{d}^1_{\text{in}}, \cdots, \mathsf{d}^N_{\text{in}}\}$. For connected graphs, The *(out-) Laplacian* matrix is $\mathsf{L} = \mathsf{D}^{\text{out}} - \mathsf{A}$, where $\mathsf{D}^{\text{out}} = \mathrm{Diag}(\mathsf{d}^1_{\text{out}}, \cdots, \mathsf{d}^N_{\text{out}}) \in \mathbb{R}^{N \times N}$. Note that $\mathsf{L}\mathbf{1}_N = \mathbf{0}$. A digraph is weight-balanced if and only if $\mathbf{1}_N^\top \mathsf{L} = \mathbf{0}$ if and only if $\mathrm{Sym}(\mathsf{L}) = (\mathsf{L} + \mathsf{L}^\top)/2$ is positive semi-definite. Based on the structure of L, at least one of the eigenvalues of L is zero and the rest of them have non-negative real parts. For a strongly connected and weight-balanced digraph, zero is a simple eigenvalue of both L and $\mathrm{Sym}(\mathsf{L})$. In this case, we order the eigenvalues of $\mathrm{Sym}(\mathsf{L})$ as $\hat{\lambda}_1 = 0 < \hat{\lambda}_2 \leq \hat{\lambda}_3 \leq \cdots \leq \hat{\lambda}_N$. We denote eignvalues of L by λ_i and sort them such that $\lambda_1 = 0$, and $\mathrm{Re}(\lambda_i) \leq \mathrm{Re}(\lambda_j)$ for any $i, j \in \{1, \dots, N\}$ and $i < j$.

III. Problem statement

Here, we formalize the problem of interest. Consider a network of N agents with single-integrator dynamics,

$$\dot{x}^i = \psi^i, \quad i \in \{1, \dots, N\},$$

where $x^i \in \mathbb{R}$ is the *agreement state* and $\psi^i \in \mathbb{R}$ is the *driving command* of agent i. Each agent $i \in \{1, \dots, N\}$ has access to a time-varying reference input signal $\mathsf{r}^i : \mathbb{R}_{\geq 0} \to \mathbb{R}$. The network interaction topology is modeled by a weighted digraph $\mathcal{G}$ that models the capability of agents to transmit information to other agents, however, the communications are subject to some fixed common transmission delay $\tau \in \mathbb{R}_{>0}$. Under the network model described above, our goal is to design a distributed algorithm that allows each agent to asymptotically track $\frac{1}{N}\sum_{j=1}^{N} \mathsf{r}^j$.

Our starting point is the continuous-time dynamic average consensus algorithm

$$\begin{aligned} \dot{v}^i &= \alpha\beta \sum_{j=1}^{N} \mathsf{a}_{ij}(x^i(t-\tau) - x^j(t-\tau)), \\ \dot{x}^i &= \dot{\mathsf{r}}^i - \alpha(x^i - \mathsf{r}^i) - \beta \sum_{j=1}^{N} \mathsf{a}_{ij}(x^i(t-\tau) - x^j(t-\tau)) - v^i, \end{aligned} \tag{3}$$

for $i \in \{1, \dots, N\}$, that is proposed in [20] and its correctness is characterized as follows when there is no communication delay in the network, i.e., $\tau = 0$.

Theorem 3.1 (Convergence of (3) over strongly connected and weight-balanced digraphs [20] without communication delay): *Assume the agent inputs satisfy* $\|\mathbf{\Pi}_N \dot{\mathsf{r}}\|_{ess} = \gamma < \infty$*. Then, for any* $\alpha, \beta \in \mathbb{R}_{>0}$*, the trajectories of the algorithm* (3) *executed on a strongly connected and weight-balanced digraph* $\mathcal{G}$ *with no communication delay, i.e.,* $\tau = 0$*, initialized at* $x^i(0), v^i(0) \in \mathbb{R}$ *with* $\sum_{i=1}^{N} v^i(0) = 0$ *are bounded and satisfy*

$$\lim_{t \to \infty} \Big| x^i(t) - \frac{1}{N} \sum_{j=1}^{N} \mathsf{r}^j(t) \Big| \leq \frac{\gamma}{\beta \hat{\lambda}_2}, \quad i \in \{1, \dots, N\}. \tag{4}$$

The rate of convergence to this error neighborhood is $\min\{\alpha, \beta Re(\lambda_2)\}$. □

Our goal in this paper is to characterize the admissible communication delay range for τ in (3) and study the effect of the communication delay on the tracking error bounds. By admissible delay range we mean values of delay for which the algorithm (3), which is a DDE system in the form of (A.19), is internally exponentially stable. Please see the Appendix for a short review of pertinent stability notion and stability analysis results for DDE systems.

IV. Convergence and stability analysis in the presence of a constant communication delay

We study the convergence and stability properties of algorithm (3) in the presence of constant communication delays, under the following assumption on its initial conditions.

Assumption 1 (Assumption on initial conditions of algorithm (3)): *The initial conditions of algorithm* (3) *satisfy* $\mathbf{x}(0), \mathbf{v}(0) \in \mathbb{R}^N$ *such that* $\sum_{i=1}^{N} v^i(0) = 0$. *Moreover, we assume that* $\mathbf{x}(t) = \mathbf{0}$ *for* $t \in [-\tau, 0)$ *and* $\mathsf{r}(t) = \mathbf{0}$ *for* $t \in [-\tau, 0)$. □

This assumption is technical and imposes no extra restriction on the class of reference inputs discussed in Theorem 3.1 for the delay free algorithm. Next, for convenience in analysis, we apply the coordinate transformation

$$\begin{bmatrix} \begin{bmatrix} q_1 \\ \mathbf{q}_{2:N} \end{bmatrix} \\ \begin{bmatrix} p_1 \\ \mathbf{p}_{2:N} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \mathfrak{r}^\top \\ \mathfrak{R}^\top \end{bmatrix} & \begin{bmatrix} \mathbf{0} \\ \alpha\,\mathfrak{R}^\top \end{bmatrix} \\ \mathbf{0} & \begin{bmatrix} \mathfrak{r}^\top \\ \mathfrak{R}^\top \end{bmatrix} \end{bmatrix} \begin{bmatrix} \mathbf{v} - \alpha\,\mathbf{\Pi}\,\mathsf{r} \\ \mathbf{x} - \bar{\mathsf{r}} \end{bmatrix}, \quad (5)$$

to represent our algorithm in the equivalent compact form

$$\dot{q}_1 = 0, \quad (6a)$$

$$\dot{\mathbf{q}}_{2:N} = -\alpha \mathbf{q}_{2:N}, \quad (6b)$$

$$\dot{p}_1 = -\alpha p_1 - q_1, \quad (6c)$$

$$\dot{\mathbf{p}}_{2:N} = -\beta \mathfrak{R}^\top \mathsf{L} \mathfrak{R}\, \mathbf{p}_{2:N}(t-\tau) + \mathfrak{R}^\top \dot{\mathsf{r}} - \mathbf{q}_{2:N}. \quad (6d)$$

Here, we used $\bar{\mathsf{r}} = \frac{1}{N}\Sigma_{j=1}^{N} \mathsf{r}^j \mathbf{1}_N$, and took into account the assumptions on the initial conditions given in Assumption 1. Given the change of variables (5), the following relationships exist between the initial conditions of (6) and those of algorithm (3) when they satisfy Assumption 1 (recall (1)),

$$q_1(0) = \mathfrak{r}^\top(\mathbf{v}(0) - \alpha\,\mathbf{\Pi}\,\mathsf{r}(0)) = \mathfrak{r}^\top \mathbf{v}(0) = 0, \quad (7a)$$

$$\|\mathbf{q}_{2:N}(0)\| = \|\alpha\,\mathbf{\Pi}(\mathbf{x}(0) - \mathsf{r}(0)) + \mathbf{v}(0)\|, \quad (7b)$$

$$p_1(0) = \mathfrak{r}^\top(\mathbf{x}(0) - \bar{\mathsf{r}}(0)), \quad (7c)$$

$$\mathbf{p}_{2:N}(t) = \mathbf{0}_{N-1}, \ t \in [-\tau, 0), \quad (7d)$$

$$\|\mathbf{p}_{2:N}(0)\| = \|\mathbf{\Pi}(\mathbf{x}(0) - \bar{\mathsf{r}}(0))\|. \quad (7e)$$

Next, remark that, for $i \in \{1, \dots, N\}$,

$$\lim_{t\to\infty} |x^i - \frac{1}{N}\sum\nolimits_{j=1}^{N} \mathsf{r}^j|^2 \leq \lim_{t\to\infty} \|\mathbf{x} - \bar{\mathsf{r}}(t)\|^2 = \lim_{t\to\infty} \Big(\|\mathbf{p}_{2:N}(t)\|^2 + |p_1(t)|^2\Big). \quad (8)$$

Here, we used the orthonormal property of matrix $[\mathfrak{r} \;\; \mathfrak{R}]$ to write $\|\mathbf{x} - \bar{\mathsf{r}}(t)\|^2 = \|\mathbf{p}\|^2 = \|\mathbf{p}_{2:N}\|^2 + |p_1|^2$. For the given initial conditions (7), we can show that the trajectories of $p_1(t)$ are exponentially vanishing with time, i.e., $\lim_{t\to\infty} |p_1(t)| = 0$. Then, to establish an upper bound on the tracking error of each agent, we need to find an upper bound on trajectories of $\mathbf{p}_{2:N}(t)$. Notice that we can consider (6d) as a DDE system with system state matrix $-\beta\,\mathfrak{R}^\top \mathsf{L}\mathfrak{R}$ and inputs $(\mathfrak{R}^\top \dot{\mathsf{r}} - \mathbf{q}_{2:N})$. Then, as reviewed in the Appendix (see (A.20)), given the initial conditions (7d) and (7e), the solution for (6d) is given by

$$\mathbf{p}_{2:N}(t) = \mathbf{\Phi}(t)\,\mathbf{p}_{2:N}(0) + \int_0^t \mathbf{\Phi}(t-\zeta)\,(\mathfrak{R}^\top \dot{\mathsf{r}}(\zeta) - \mathbf{q}_{2:N}(\zeta))\mathrm{d}\zeta, \quad (9)$$

where $\Phi(t-\zeta) = \sum_{k=-\infty}^{k=\infty} \mathrm{e}^{\mathbf{S}_k(t-\zeta)}\mathbf{C}_k$ with $\mathbf{S}_k = \frac{1}{\tau}\mathrm{W}_k(-\beta\,\mathfrak{R}^\top \mathsf{L}\mathfrak{R}\tau)$, and each coefficient $\mathbf{C}_k$ depending on τ and $-\beta\,\mathfrak{R}^\top \mathsf{L}\mathfrak{R}$. Using (9) we can write

$$\|\mathbf{p}_{2:N}(t)\| \leq \|\mathbf{\Phi}(t)\,\mathbf{p}_{2:N}(0)\| + \int_0^t \|\mathbf{\Phi}(t-\zeta)\|\,(\|\mathbf{\Pi}\dot{\mathsf{r}}(\zeta)\| + \|\mathbf{q}_{2:N}(\zeta)\|)\mathrm{d}\zeta. \quad (10)$$

Here, we used (1) to write $\|\mathfrak{R}^\top\,\dot{\mathsf{r}}\| = \|\mathbf{\Pi}\,\dot{\mathsf{r}}(\zeta)\|$. To study the ultimate behavior of $\|\mathbf{p}_{2:N}\|$, we start our analysis by characterizing an admissible range for delay τ such that the zero-input dynamics of DDE system (6d), i.e.,

$$\dot{\mathbf{p}}_{2:N}(t) = -\beta\,\mathfrak{R}^\top \mathsf{L}\mathfrak{R}\,\mathbf{p}_{2:N}(t-\tau), \quad (11)$$

is exponentially stable. Note that (11) is the well-known static Laplacian average consensus algorithm with zero eigenvalue separated. For connected graphs, [8] uses the Nyquist criterion to characterize the admissible delay bound. Here, we use characteristic polynomial analysis of [28, Theorem 1] for linear delay systems to obtain the delay bounds for strongly connected and weight-balanced digraphs.

Lemma 4.1 (Admissible range of τ for (11) over strongly connected and weight-balanced digraphs): *Let the communication graph be strongly connected and weight-balanced. Then, for any* $\tau \in [0, \bar{\tau})$ *the zero-input dynamics* (11) *exponentially stable if and only if*

$$\bar{\tau} = \min\Big\{\tau \in \mathbb{R}_{>0} \,\Big|\, \tau = \frac{|atan(\frac{Re(\lambda_i)}{Im(\lambda_i)})|}{\beta\,|\lambda_i|},\ i \in \{2, \cdots, N\}\Big\}, \quad (12)$$

where λ_i, $i \in \{2, \cdots, N\}$ *are non-zero eigenvalues of* L. *Moreover, if the graph is connected this admissible range is given by* $\bar{\tau} = \frac{\pi}{2\,\beta\lambda_N}$.

Sketch of the proof: proof relies on invoking the results of Theorem A.2. □

In the following, using $\alpha-$stability type analysis, we present a result that characterizes the rate of convergence of the solutions of (11) in terms of a given admissible delay τ and network parameters. Using this result, we obtain an upper bound on $\|\mathbf{\Phi}(t-\zeta)\|$ of (9).

Lemma 4.2 (An estimate on the worst rate of convergence of the zero-input dynamics (11)): *Let the communication graph be strongly connected and weight-balanced and* $\{\lambda_i\}_{i=2}^{N}$ *be non-zero eigenvalues of* L. *Given an admissible*

$\tau \in [0, \bar{\tau})$ *where* $\bar{\tau}$ *is given in* (12), *an estimate on the rate of convergence* ρ_τ, *as defined in* (A.21), *for trajectories of* (11) *is given by*

$$\rho_\tau = -\frac{1}{\tau}\max\left\{\bar{\rho} \in \mathbb{R}_{<0} \,\middle|\, \bar{\rho} = \mathrm{Re}(W_0(-\beta\lambda_i\,\tau)),\right. \\ \left. i \in \{2, \cdots, N\}\right\}. \quad (13)$$

Proof: Our proof is based on the α-stability argument for delay systems. For a given τ, we use $\mathbf{z}(t) = \mathrm{e}^{\psi t}\,\mathbf{p}_{2:N}(t)$, $\psi \in \mathbb{R}_{>0}$ to write (11) as

$$\dot{\mathbf{z}} = \psi\,\mathrm{e}^{\psi t}\,\mathbf{p}_{2:N} + \mathrm{e}^{\psi t}\,\dot{\mathbf{p}}_{2:N} = \psi\,\mathbf{z} - \mathrm{e}^{\psi\tau}\beta\mathfrak{R}^\top\mathsf{L}\mathfrak{R}\,\mathbf{z}(t-\tau). \quad (14)$$

The characteristic equation of (14) is

$$\mathcal{F} = \det\Big(s\,\mathbf{I} - \psi\,\mathbf{I} + \beta\mathfrak{R}^\top\mathsf{L}\mathfrak{R}\,\mathrm{e}^{\psi\tau}\mathrm{e}^{-s\tau}\Big) = \\ \prod_{i=2}^{N}(s - \psi + \beta\lambda_i\,\mathrm{e}^{\psi\tau}\mathrm{e}^{-s\tau}). \quad (15)$$

Next, for the given τ, we find the largest ψ such that all the eigenvalues of the characteristic polynomial are stable. To this end, we invoke the continuity result of Lemma (A.1) and determine for what values of ψ, the eigenvalues will cross the imaginary axis. This value serves as an upper-bound on ranges of ψ which system (14) is exponentially stable for. From the characteristic equation (15), we obtain

$$s - \psi + \beta\lambda_i\,\mathrm{e}^{\psi\tau}\mathrm{e}^{-s\tau} = 0 \Rightarrow (s-\psi)\tau\,\mathrm{e}^{(s-\psi)\tau} = -\beta\lambda_i\,\tau \\ \Rightarrow s - \psi = \frac{1}{\tau}W(-\beta\lambda_i\,\tau). \quad (16)$$

According to Theorem A.1, system (14) is exponentially stable if and only if, for all $i \in \{2, \cdots, N\}$,

$$\max\{\mathrm{Re}(s)\} = \psi + \frac{1}{\tau}\max\{\mathrm{Re}(W(-\beta\lambda_i\,\tau))\} \\ = \psi + \frac{1}{\tau}\,\mathrm{Re}(W_0(-\beta\lambda_i\,\tau)) < 0.$$

Here, we used Lemma 2.1. Thus, we can choose the largest admissible ψ under which (14) is exponentially stable as

$$\psi = -\frac{1}{\tau}\max\{\mathrm{Re}(W_0(-\beta\lambda_i\,\tau))\}_{i=2}^{N} - \epsilon, \quad (17)$$

where $0 < \epsilon << 1$ is an infinitesimally small positive scalar. Recall that $\|\mathbf{p}_{2:N}(t)\| = \mathrm{e}^{-\psi t}\|\mathbf{z}(t)\|$, then, given that the system (14) is exponentially stable for (17), we can conclude that an estimate for ρ_τ in (A.21) for (11), is given by $-\frac{1}{\tau}\max\{\mathrm{Re}(W_0(-\beta\lambda_i\,\tau))\}_{i=2}^{N}$. ■

For delays in admissible range characterized in Lemma 4.1, the zero-input dynamics is exponentially stable and as a result we can establish and characterize an upper bound for $\|\mathbf{\Phi}(t)\|$. The next result describes this upper bound and uses it to establish the ultimate tracking bound for the distributed average consensus algorithm (3)

Theorem 4.1 (Convergence of (3) over strongly connected and weight-balanced digraphs [20]): *Assume the agent inputs satisfy* $\|\mathbf{\Pi}_N\,\dot{\mathbf{r}}\|_{ess} = \gamma < \infty$. *Then, for any* $\alpha, \beta > 0$, *the trajectories of the algorithm* (3) *executed on a strongly connected and weight-balanced digraph* $\mathcal{G}$ *with communication delay in* $\tau \in [0, \bar{\tau})$ *where* $\bar{\tau}$ *is given in* (12), *initialized such that Assumption 1 is satisfied, are bounded and satisfy*

$$\lim_{t\to\infty}\Big|x^i(t) - \frac{1}{N}\sum\nolimits_{j=1}^{N}\mathsf{r}^j(t)\Big| \leq \kappa_\tau\frac{\gamma}{\rho_\tau},\; i \in \{1, \ldots, N\}, \quad (18)$$

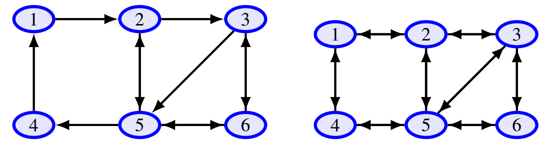
where $\rho_\tau \in \mathbb{R}_{>0}$ *satisfies* (13) *and* $\kappa_\tau \in \mathbb{R}_{>0}$ *satisfies* $\|\mathbf{\Phi}(t)\| \leq \kappa_\tau e^{-\rho_\tau t}$ *for* $t \in \mathbb{R}_{\geq 0}$. *The rate of convergence to this error neighborhood is* $\min\{\alpha, \rho_\tau\}$. □

When local dynamic reference signals across the network are only offset from one another by static constant values, $\gamma = 0$ in $\|\mathbf{\Pi}_N\dot{\mathbf{r}}\|_{\mathrm{ess}} = \gamma$. Therefore, in this case the algorithm (3) when the communication delay satisfies (12) asymptotically tracks the desired average value with zero error. It is also interesting to observe that $\lim_{\tau\to 0}\rho_\tau = \lim_{\tau\to 0} -\frac{1}{\tau}\max\{\mathrm{Re}(W_0(-\beta\lambda_i\,\tau))\}_{i=2}^{N} = -\max\{\mathrm{Re}(-\beta\lambda_i)\}_{i=2}^{N} = \beta\,\mathrm{Re}(\lambda_2)$ (recall (2)). This means that as τ approaches 0, the rate of convergence of the dynamic consensus algorithm converges to the rate of the convergence established in Theorem 3 for the dynamics with no communication delay.

We close this section by establishing a conservative upper bound on admissible ranges of τ in terms of the maximum degree of the communication topology when the topology is strongly connected and weight-balanced. Earlier results connecting delay bounds to the degree of network in the consensus algorithms only discuss connected graphs.

Lemma 4.3 (Admissible range of τ for (11) in terms of maximum degree of the digraph): *Let the communication graph be strongly connected and weight-balanced. Then, for any* $\tau \in [0, \bar{\tau})$ *the zero-input dynamics* (11) *is exponentially stable if* $\bar{\tau} \leq \frac{1}{2\,\beta\,\mathsf{d}^{\max}}$. *For connected graphs, we can obtain a less conservative bound as* $\bar{\tau} \leq \frac{\pi}{4\,\beta\,\mathsf{d}^{\max}}$. □

It is interesting to observe the inverse relation between the maximum admissible delay and the maximum degree of the communication topology. Such an adverse effect from higher maximum degree on delay bound is aligned with the intuition. One can expect that the more links to arrive an some agents of the network, the more susceptible the convergence of the algorithm will be to the larger delays.



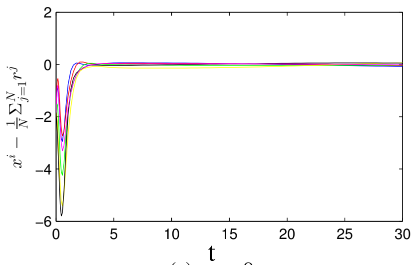
(a) (b)

Fig. 1: The communication topologies used in simulations. The weights are 0 and 1.

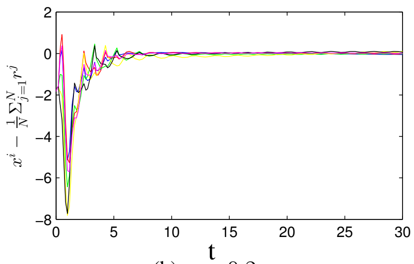
V. Numerical Simulations

In this section, we analyze and demonstrate the performance of the algorithm (3) in the presence of communication delay over networked topologies depicted in Figure 1. and reference signals

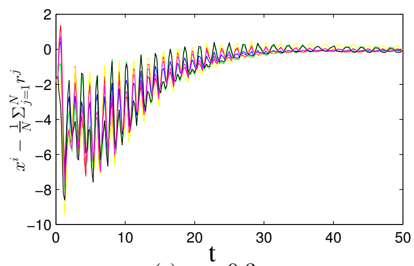
$$\mathsf{r}^1(t) = 2\sin(0.1t) + \tfrac{1}{(t+1)} + 2,\; \mathsf{r}^2(t) = 2\sin(0.1\,t) + \mathrm{e}^{-6t} + 4,$$



(a) $\tau = 0$



(b) $\tau = 0.2\,s$



(c) $\tau = 0.3\,s$

Fig. 2: Time history of the tracking error $x^i(t) - \frac{1}{N}\sum_{j=1}^{N} \mathsf{r}^j(t)$ at each agent $i \in \{1, \ldots, N\}$ when algorithm (3) executed over the network topology depicted in Fig. 1.(a) in the absence and presence of communication delay.

$$\mathsf{r}^3(t) = 2\sin(0.1t) + \frac{1}{(t+4)^5} + 1,\ \mathsf{r}^4(t) = 2\cos(0.1\,t) + \mathrm{e}^{-8t} - 2,$$
$$\mathsf{r}^5(t) = -2\cos(0.1\,t) + \frac{t^2}{t^4+t+6},\ \mathsf{r}^6(t) = 2\sin(0.1\,t) + 3.$$

For $\beta = 1$, the maximum admissible delay bound for the topologies depicted in Fig. 1.(a) and Fig. 1.(b) are, respectively, $\bar{\tau}_a = 0.52$ seconds and $\bar{\tau}_b = 0.30$ seconds. It is interesting to observe the adverse effect of a higher maximum degree of the communication topology on the upper bound of the admissible delay bound (see Lemma 4.3). Notice that for the network in Fig. 1.(a) the maximum degree of the graph is $\mathsf{d}^{\max} = 3$ while for the network in Fig. 1.(b) we have $\mathsf{d}^{\max} = 4$.

Fig. 2 shows trajectories of algorithm (3) over the network topology of Fig. 1.(a) for fixed values of $\alpha = 4$, $\beta = 1$ and (a) $\tau = 0$, (b) $\tau = 0.2s$, (c) $\tau = 0.3s$. As this figure shows, by increasing time delay, the rate of convergence of the algorithm decreases. Therefore, as can be predicted from (18), the tracking error increases. The least convergence rate for each of the aforementioned simulation scenarios (a) and (b) are given by, respectively (a) $\min(\alpha = 4, \rho_\tau = 1.11) = 1.11$ and (b) $\min(\alpha = 4, \rho_\tau = 0.77) = 0.77$.

VI. Conclusions

We have studied the multi-agent dynamic average consensus problem over networks when the communication between agents is subject to a fixed common delay. Our study included characterizing admissible delay range for a previously developed delay-free continuous-time dynamic average consensus algorithm which is known to converge exponentially to a small neighborhood of the network's inputs average. We also studied the effect of communication delay on the rate of convergence and the tracking error and obtain upper bounds for them based on the value of the communication delay and the network's and the algorithm's parameters. Future work will be devoted to investigating the effect of uncommon communication delay on the algorithms stability and convergence. We will also expand our results for switching graphs and also discrete-time implementation of the algorithm.

Appendix

A Review of Stability of Time-delay systems of retarded type

Here, for the convenience of the reader, we briefly review some relevant properties of the linear time-delay systems of retarded type described by the following DDE

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t-\tau) + \mathbf{B}\mathbf{u}(t), \tag{A.19}$$
$$\mathbf{x}(t) = \mathbf{g}(t), \quad t \in [-\tau, 0); \quad \mathbf{x}(t) = \mathbf{x}_0, \quad t = 0,$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state variable at time t, $\mathbf{A} \in \mathbb{R}^{n\times n}$ is the system matrix, $\mathbf{g}(t)$ and $\mathbf{x}_0$ are, respectively, a specified pre-shape function and an initial state, and $\tau \in \mathbb{R}_{>0}$ represents the *time-delay*. A discontinuity is permitted at $t = 0$, i.e., $\mathbf{g(t)}(0^-) \neq \mathbf{x}_0$ [29]. In this paper we consider a special class of DDE system (A.19) which satisfies the following assumption.

Assumption 2: We assume that pre-shape initial condition function satisfies $\mathbf{g(t)} \equiv \mathbf{0}$ for $t \in [-\tau, 0)$. □

Then, under assumption 2, following [24], the trajectories of the DDE system (A.19) for $t \in \mathbb{R}_{\geq 0}$ are given by

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\,\mathbf{x}_0 + \int_0^t \mathbf{\Phi}(t-\zeta)\,\mathbf{B}\mathbf{u}(\zeta)\,\mathrm{d}\zeta, \tag{A.20}$$

where $\Phi(t-\zeta) = \sum_{k=-\infty}^{k=\infty} \mathrm{e}^{\mathbf{S}_k(t-\zeta)}\mathbf{C}_k$ with $\mathbf{S}_k = \frac{1}{\tau}\mathrm{W}_k(-\mathbf{A}\tau)$ and $\mathrm{W}_k(.)$ being the k^{th} Lambert function branch. Here, each coefficient $\mathbf{C}_k$ depends on τ and $\mathbf{A}$ and the method to compute them can be found in [24].

The notion of exponential stability is defined similar to linear time invariant system, i.e.,

Definition A.1 (Exponential internal stability of the linear retard system (A.19), see e.g., [23]): *The trivial solution* $\mathbf{x} \equiv \mathbf{0}$ *of* (A.19) *is said to be exponentially stable if there exists a* $\kappa_\tau \in \mathbb{R}_{>0}$ *and an* $\rho_\tau \in \mathbb{R}_{>0}$ *such that for the given initial condition in* (A.19) *the solution satisfies the inequality:*

$$\|\mathbf{x}(t)\| \leq \kappa_\tau e^{-\rho_\tau t} \underset{t\in[-\tau,0]}{sup} \|\mathbf{x}(t)\|, \quad t \in \mathbb{R}_{\geq 0}. \tag{A.21}$$

□

Let $\{\mu_i\}_{i=1}^n$ be eigenvalues of $\mathbf{A}$. The exponential stability of (A.19) in terms of the roots of its characteristic equation $\mathcal{F} : \mathbb{C} \to \mathbb{C}$ given by

$$\mathcal{F}(s) = \det\Big(s\mathbf{I}_n - \mathbf{A}\mathrm{e}^{-\tau s}\Big) = \prod_{i=1}^{n}(s - \mu_i \mathrm{e}^{-\tau s}), \tag{A.22}$$

is characterized as follows.

Theorem A.1 (Exponential stability of DDE system (A.19) [23]): *The DDE system* (A.19) *is exponentially stable if and only if*

$$\{s \in \mathbb{C} \,|\, \mathrm{Re}(s) \geq 0, \mathcal{F}(s) = 0\} = \emptyset, \qquad \text{(A.23)}$$

where $\emptyset$ *is the empty set.* □

Due to its form, the characteristic equation associated to $\mathcal{F}$ is transcendental and has an infinite number of roots in the complex plane. Next, we review a result that facilitates characterizing *admissible range* for delay τ in which the exponential stability of system (A.19) is preserved.

Lemma A.1 (the continuity stability property over admissible ranges of τ in (A.19), see Proposition 3.1 [23]): *Consider the system* (A.19) *and let*

$$\mathcal{S}(r) = \big\{\mathbf{A} \in \mathbb{R}^{n\times n} | \text{(A.19) is exponentially stable for } \tau = r\big\}.$$

Then the following properties hold:

(a) *if* $\mathbf{A} \in \mathcal{S}(0)$*, then there exists* $\epsilon > 0$ *sufficiently small such that* $\mathbf{A} \in \mathcal{S}(h)$ *for all* $h \in [0, \epsilon]$.
(b) *if* $\mathbf{A} \in \mathcal{S}(0)$*, and if there exists a* $\tau_{unstable}$ *for which system* (A.19) *with* $\tau = \tau_{unstable}$ *is not stable with a strictly unstable root, then there exists an* $\epsilon \in (0, \tau_{unstable})$ *such that*
   - $\mathbf{A} \in \mathcal{S}(h)$ *for all* $h \in [0, \epsilon)$*, and*
   - *for* $h = \epsilon$ *the corresponding characteristic equation* (A.22) *has roots on the imaginary axis.*

From this lemma one can see that a tight upper bound on admissible ranges of τ can be obtained by finding the minimum value of τ such that the characteristic equation (A.22) has eigenvalues on the imaginary axis. Using such an approach the admissible ranges of τ for the zero input dynamics of system (A.19) is given in [28, Theorem 1] reviewed below.

Theorem A.2 (Admissible range for delay τ in (A.19) [28]): *Let the zero-input dynamics of DDE system* (A.19) *with* $\tau = 0$ *be exponentially stable. Then, the zero-input dynamics is exponentially stable for any* $\tau \in [0, \bar{\tau})$*, where* $\bar{\tau}$ *is given by*

$$\bar{\tau} = \min\Big\{\tau \in \mathbb{R}_{>0} \,\Big|\, \tau = \frac{|atan(\frac{\mathrm{Re}(\mu_i)}{\mathrm{Im}(\mu_i)})|}{|\mu_i|}, \quad i \in \{1, \dots, n\}\Big\}.$$

Consequently, when $\mathbf{A}$ *is also a symmetric negative definite matrix, then* $\bar{\tau} = \frac{\pi}{2\,\mu_{max}}$*, where* μ_{max} *is the maximum eigenvalue of* $-\mathbf{A}$. □

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