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Distributed Minimum Energy Leader-Follower Algorithm for Multi-Agent Systems with an Active Non-Homogenous Leader

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Abstract— **In this paper, we consider a leader-follower problem for a group of homogeneous linear time invariant (LTI) follower agents that are interacting over a directed acyclic graph. In our problem of interest, only a subset of the follower agents has access to the state of the leader in specific sampling times. The dynamics of the leader that generates its states is unknown to the followers. For interaction topologies in which the leader is a global sink in the graph, we propose a distributed algorithm that allows the agents to arrive at the sampled state of the leader before the next sample arrives. We prove that the control input to take the followers from one sampled state to the next one is minimum energy for all the followers. We also show that after the first sampling epoch, the states of all the follower agents are synchronized with each other. We demonstrate the application of our proposed algorithm for two leader-follower problems for mobile agents. Our first example shows the application of our algorithm in control of unicycle robots in a formation motion. In the second example, we demonstrate the use of our algorithm for reference state tracking via a group of second order integrator followers with bounded control. In this example, we show that the properties of our proposed leader-follower algorithm allow us to design the arrival times at the reference states in such a way that the input bounds of the agents never get violated.**

I. Introduction

Synchronization of multi-agent systems (MASs) is an important component of many cooperative control problems, such as rendezvous [1], formation control [2], flocking control [3], containment control [4] and sensor networks [5]. Synchronization problems are usually categorized into leaderless and leader-following problems. In the leaderless synchronization, which is closely related to the consensus problem, the agents aim to reach to a static or dynamic agreement on a common value [6], [7], [8]. On the other hand, in the leader-following synchronization, agents aim to make the agreement on the states generated by a leader. In this paper, we focus on the design of a distributed leader-follower algorithm when the only information available about the leader is its sampled state, which is only available to a subset of follower agents.

Literature review: The leader-following algorithms for single integrator and double integrator dynamics are presented in [9] and for linear time invariant (LTI) systems are proposed in [10] and [11]. However, the leader in these references is passive, i.e., the leader has a zero-input LTI dynamics. [12] and [13], respectively, develop distributed controllers for single and double integrator follower groups to track an active LTI leader. But they both assume the leader's control input is available to all the followers. [14] proposes a leader-following algorithm for a homogeneous LTI follower and leader agents in which the unknown input of the leader is bounded and is not available to any follower. This algorithm has a sliding mode structure and suffers from the well-known undesirable chattering behavior. We recall that from a practical perspective, chattering is undesirable and leads to excessive control energy expenditure [15]. [16] is the recent result for the leader-following problem, which is based on the result of [14] and develop a distributed observer designed to estimate the leader's state for each follower. Then, the output synchronization of heterogeneous leader-follower linear systems is achieved by optimal local tracking of the output of the observer. We note that in both [14] and [16], the active leader is restricted to have a bounded input. For the works mentioned so far, the settling time to the leader following manner is infinity. [17] and [18] propose the finite time leader following algorithm in which the settling time is upper bounded by a estimation. [19] introduces the specified-time synchronization control for the leaderless MASs in which one can determine the settling time exactly in advance.

Statement of contributions: The objective of this paper is to design a leader-follower algorithm, which steers a group of follower agents with LTI dynamics to be at the sampled states of a leader agent at finite specified times. We make no assumptions about the structural form of the leader's dynamics or its input. The sampled states of the leader can be the states of a physical system or a set of desired reference states of a virtual leader. We assume that the only information available about the dynamics of the leader is its sampled states, which is known only to a subset of the followers at the sampling times. Inspired by the classical optimal control results, we propose a distributed minimum energy control strategy to solve the leader-follower problem for networks that the interaction topology of the followers plus the leader is an acyclic digraph with the leader as the global sink. This algorithm not only results in a leader-following behavior, but also it makes the states and inputs of the agents to become fully synchronized after the first sampling epoch. We demonstrate the effectiveness of our proposed leader-follower algorithm for two application examples for mobile agents. In the first example, we show the application of our proposed algorithm in a leader-following task under a specific formation structure for a group of unicycle robots. In this example, the leader's dynamics is nonlinear while

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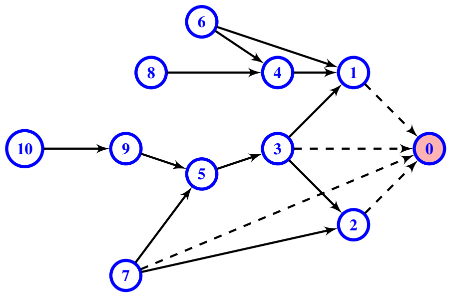

Fig. 1: A leader-follower network. The interaction topology of the follower agents, $\mathcal{G}$, shown via the network with solid edges, is an acyclic digraph. Agent 0 is the leader. The edges of $\mathcal{G}_l$ is shown by the dashed arrow. Here, the leader is the global sink of the $\mathcal{G} \cup \mathcal{G}_l$, therefore, its information reaches all the agents in an explicit or implicit manner.

the dynamics of the followers are feedback linearized. In the second example, we demonstrate the use of our algorithm for reference state tracking via a group of second order integrator followers with bounded control. Using the intrinsic properties of our leader-following algorithm, in this example, we show that the arrival times at the reference states can be designed in such a way that the inputs of the agents stay within the pre-specified saturation bounds.

Organization: The rest of this parer is organized as follows. Section II gathers basic notation and graph-theoretic terminology and notions. Section III formulates the leader-follower problem of a group of homogeneous LTI followers with an active non-homogeneous leader. Section IV proposes our distributed leader-follower algorithm. Section V demonstrates our results using two numerical application examples. Section VI concludes the results of this paper. Due to the space limitations, the proofs are omitted and will appear elsewhere.

II. Notations and Preliminaries

Notation: We let $\mathbb{R}$, $\mathbb{R}_{>0}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Z}$, and $\mathbb{Z}_{\geq 0}$ denote the set of real, positive real, non-negative real, integer, and non-negative integer numbers, respectively. The transpose of a matrix $\mathbf{A} \in \mathbb{R}^{n\times m}$ is $\mathbf{A}^\top$.

Graph theoretic notations and definitions: Here we review our graph related notations and relevant definitions following [20]. A *digraph*, is a triplet $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathsf{A})$, where $\mathcal{V} = \{1, \dots, N\}$ is the *node set* and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the *edge set*, and $\mathsf{A} = [\mathsf{a}_{ij}] \in \mathbb{R}^{N\times N}$ is the *adjacency* matrix of the graph defined according to $\mathsf{a}_{ij} = 1$ if $(i,j) \in \mathcal{E}$ and $\mathsf{a}_{ij} = 0$, otherwise. An edge (i,j) from i to j means that agent j can send information to agent i (or equivalently, agent i can get information from agent j). Here, i is called an *in-neighbor* of j and j is called an *out-neighbor* of i. A *directed path* is a sequence of nodes connected by edges. A directed path that starts and ends at the same node and all other nodes on the path are distinct is called a *cycle*. A digraph without cycles is called *directed acyclic graph*. The *out-degree* of a node i is $\mathsf{d}^i_{\text{out}} = \Sigma^N_{j=1} \mathsf{a}_{ij}$. The out-degree matrix of a graph is $\mathsf{D}_{\text{out}} = \text{Diag}(\mathsf{d}^1_{\text{out}}, \mathsf{d}^2_{\text{out}}, \cdots, \mathsf{d}^N_{\text{out}})$. We denote the set of in-neighbors of an agent i by $\mathcal{N}^i_{\text{in}}$ and the out-neighbors of agent i by $\mathcal{N}^i_{\text{out}}$. A node $i \in \mathcal{V}$ is called a *global sink* of $\mathcal{G}$ if it outdegree $\mathsf{d}^i_{\text{out}} = 0$ and for every node $j \in \mathcal{V}$ there is a path from j to i.

III. Problem definition

In this section we formalize our problem of interest. We consider a group of N MAS whose dynamics is described by

$$\dot{\mathbf{x}}^i(t) = \mathbf{A}\,\mathbf{x}^i(t) + \mathbf{B}\,\mathbf{u}^i(t), \quad i \in \mathcal{V} = \{1, \cdots, N\}, \quad (1)$$

where $\mathbf{x}^i \in \mathbb{R}^n$ is the state vector and $\mathbf{u}^i \in \mathbb{R}^m$ is the control vector. These agents aim to follow a dynamic signal $\mathbf{x}^0(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$. This signal can be a dynamic reference signal of a virtual leader or the state of an active physical leader with nonlinear dynamics

$$\dot{\mathbf{x}}^0(t) = f^0(\mathbf{x}^0(t), \mathbf{u}^0(t), t). \quad (2)$$

in which the control vector $\mathbf{u}^0 \in \mathbb{R}^{m^0}$ is unknown. The interaction topology between the follower agents is described by a acyclic digraph, denoted by $\mathcal{G}$. A subset of agents in $\mathcal{G}$ has access to $\mathbf{x}^0(t)$ at the sampling times t_k, $k \in \mathbb{Z}_{\geq 0}$. Throughout the paper we assume that $T_k = t_{k+1} - t_k \in \mathbb{R}_{>0}$ for any $k \in \mathbb{Z}_{\geq 0}$ with $t_0 = 0$. Moreover, we let $\mathcal{N}^0_{\text{in}}$ be the subset of the agents in $\mathcal{G}$ that are connected to the leader. We let $\mathcal{G}_l$ be the digraph consisted of the leader and $\mathcal{N}^0_{\text{in}}$ and the directed edges connecting $\mathcal{N}^0_{\text{in}}$ to the leader. In what follows, we assume that the leader is the global sink of the $\mathcal{G} \cup \mathcal{G}_l$, so that its information reaches all the agents in an explicit or implicit manner (see Fig. 1 for an example).

The objective of this paper is to design a distributed control rule for the input vector $\mathbf{u}^i(t)$ of each agent such that they can cooperatively steer the group to be at the state $x^0(t_k)$ of the leader in time before the next sampling time t_{k+1}, i.e.,

$$\mathbf{x}^i(t_{k+1}) = \mathbf{x}^0(t_k), \quad i \in \{1, \dots, N\}.$$

Note that the agents have no information about the dynamics that creates the sampled states $x^0(t_k)$. Here, we assume that agents' dynamics (1) is controllable. Recall that if a linear system is controllable, there always exists a control to move the state of the system from any point in the state space to any other point in finite time.

IV. Main result

In this section, we develop a novel distributed solution to solve the leader-follower problem stated in Section III. We start by using a classical optimal control result to make the following statement. To present this result, we recall that

$$\mathbf{G}(t) = \int_0^t \mathrm{e}^{\mathbf{A}(t-\tau)}\,\mathbf{B}\mathbf{B}^\top\,\mathrm{e}^{\mathbf{A}^\top(t-\tau)} d\tau, \quad (3)$$

is the controllability Gramian of $(\mathbf{A}, \mathbf{B})$ for any finite time $t \in \mathbb{R}_{>0}$. Since $(\mathbf{A}, \mathbf{B})$ is controllable, $\mathbf{G}(t)$ is full rank and invertible at each time $t \in \mathbb{R}_{>0}$.

Lemma IV.1. Consider a leader-follower problem where each agent' dynamics is given by (1). Suppose i is a follower agent in $\mathcal{G}$ that has access to $\mathbf{x}^0(t)$ of the leader at each sampling time t_k, $k \in \mathbb{Z}_{\geq 0}$, i.e., $i \in \mathcal{N}_{\text{in}}^0$. Starting at an initial condition $\mathbf{x}^i(t_0) \in \mathbb{R}^m$ and $\mathbf{u}^i(t_0) = \mathbf{0}$, for any $i \in \mathcal{N}_{\text{in}}^0$ let

$$\mathbf{u}^i(t) = \mathbf{B}^\top \mathrm{e}^{\mathbf{A}^\top (t_{k+1}-t)}\, \mathbf{G}_k^{-1}(\mathbf{x}^0(t_k) - \mathrm{e}^{\mathbf{A}T_k}\, \mathbf{x}^i(t_k)), \quad t \in (t_k, t_{k+1}], \tag{4}$$

where $T_k = t_{k+1} - t_k \in \mathbb{R}_{>0}$, and

$$\mathbf{G}_k = \mathbf{G}(T_k) = \int_0^{T_k} \mathrm{e}^{\mathbf{A}(T_k-\tau)}\, \mathbf{B}\mathbf{B}^\top \mathrm{e}^{\mathbf{A}^\top (T_k-\tau)} d\tau. \tag{5}$$

Then, for every $i \in \mathcal{N}_{\text{in}}^0$ we have $x^i(t_{k+1}) = x^0(t_k)$ for all $k \in \mathbb{Z}_{\geq 0}$. Moreover, at each time interval $[t_k, t_{k+1}]$, the control input $\mathbf{u}^i(t)$ of $i \in \mathcal{N}_{\text{in}}^0$ satisfies

$$\mathbf{u}^i(t) = \operatorname{argmin} \int_{t_k}^{t_{k+1}} \mathbf{u}^i(\tau)^\top \mathbf{u}^i(\tau) \mathrm{d}\tau, \quad s.t. \tag{6a}$$

$$\dot{\mathbf{x}}^i(t) = \mathbf{A}\, \mathbf{x}^i(t) + \mathbf{B}\, \mathbf{u}^i(t), \tag{6b}$$

$$\mathbf{x}^i(t_k) = \mathbf{x}^i(t_k), \quad \mathbf{x}^i(t_{k+1}) = \mathbf{x}^0(t_k). \tag{6c}$$

Proof: The proof follows from the classical finite time minimum energy optimal control design [21, page 138].

Recall here that $\mathbf{G}_k$ in (3) is the controllability Gramian of $(\mathbf{A}, \mathbf{B})$. Since $(\mathbf{A}, \mathbf{B})$ is controllable, the matrix $\mathbf{G}_k$ is invertible.

Inspired with the classical optimal control result in Lemma IV.1, in the following we propose a distributed cooperative control law that allows follower agents which do not have direct access to the leader's sampled state to also satisfy

$$\mathbf{x}^i(t_{k+1}) = \mathbf{x}^0(t_k), \quad i \in \{1, \dots, N\}.$$

To present our results we first introduce some notations. We denote the adjacency matrix and out-degree matrix of the followers' interaction topology $\mathcal{G}$, respectively, by $\mathbf{A} = [\mathsf{a}_{ij}]$ and $\mathbf{D}_{\text{out}} = \mathrm{Diag}(\mathsf{d}_{\text{out}}^1, \mathsf{d}_{\text{out}}^2, \cdots, \mathsf{d}_{\text{out}}^N)$. We let

$$\mathit{1}^i = \begin{cases} 1, & i \in \mathcal{N}_{\text{in}}^0, \\ 0, & \text{otherwise}, \end{cases} \tag{7}$$

be the indicator operator that defines the state of connectivity of agent i of $\mathcal{G}$ to the leader. We also define

$$\bar{\mathbf{G}}_k(t) = \int_{t_k}^{t} \mathrm{e}^{\mathbf{A}(t-\tau)}\, \mathbf{B}\mathbf{B}^\top \mathrm{e}^{\mathbf{A}^\top (t_{k+1}-\tau)} d\tau, \quad t \in [t_k, t_{k+1}]. \tag{8}$$

We notice that $\bar{\mathbf{G}}_k(t) = \mathbf{G}(t - t_k)\mathrm{e}^{\mathbf{A}^\top (t_{k+1}-t)}$, where $\mathbf{G}$ is the controllability gramian (3). Therefore at each finite time $t \in (t_k, t_{k+1}]$, by virtue of controllability of $(\mathbf{A}, \mathbf{B})$, $\bar{\mathbf{G}}_k(t)$ is invertible.

With the proper notations at hand, we present our distributed solution to solve our leader-follower problem of interest as follows.

Theorem IV.1. Consider a leader-follower problem where the follower agents' dynamics are given by (1). Suppose the leader's time-varying state is $\mathbf{x}^0 : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$. Let the network topology be such that $\mathcal{G} \cup \mathcal{G}_l$ an acyclic digraph with 0 as the global sink. Suppose every follower agent $i \in \mathcal{N}_{\text{in}}^0$ has access to $\mathbf{x}_0(t)$ at each sampling time t_k, $k \in \mathbb{Z}_{\geq 0}$. Let $\mathbf{P}(t) = \bar{\mathbf{G}}_k^{-1}(t)$ for $t \in (t_k, t_{k+1}]$. Starting at an initial condition $\mathbf{x}^i(t_0) \in \mathbb{R}^n$ and $\mathbf{u}^i(t_0) = \mathbf{0}$, let for $t \in (t_k, t_{k+1}]$

$$\begin{aligned}\mathbf{u}^i(t) = &\omega_l \left(\mathbf{B}^\top \mathrm{e}^{\mathbf{A}^\top (t_{k+1}-t)}\, \mathbf{G}_k^{-1}(\mathbf{x}^0(t_k) - \mathrm{e}^{\mathbf{A}T_k}\, \mathbf{x}^i(t_k))\right) + \\ &\omega_f \Big(\mathbf{B}^\top \mathrm{e}^{\mathbf{A}^\top (t_{k+1}-t)}\, \mathbf{P}(t) \times \\ &\sum_{j=1}^{N} \mathsf{a}_{ij}(\mathbf{x}^j(t) - \mathrm{e}^{\mathbf{A}(t-t_k)}\, \mathbf{x}^j(t_k)) + \\ &\mathbf{B}^\top \mathrm{e}^{\mathbf{A}^\top (t_{k+1}-t)}\, \mathbf{G}_k^{-1}\, \mathrm{e}^{\mathbf{A}T_k} \sum_{j=1}^{N} \mathsf{a}_{ij}(\mathbf{x}^j(t_k) - \mathbf{x}^i(t_k))\Big),\end{aligned} \tag{9}$$

where $\omega_l = \frac{\mathit{1}^i}{\mathit{1}^i + \mathsf{d}_{\text{out}}^i}$, $\omega_f = \frac{1}{\mathit{1}^i + \mathsf{d}_{\text{out}}^i}$. Then, the followings hold for $t \in \mathbb{R}_{\geq 0}$ and $k \in \mathbb{Z}_{\geq 0}$:

(a) $x^i(t_{k+1}) = x^0(t_k)$, $i \in \{1, \dots, N\}$;

(b) the trajectory of every follower $i \in \{1, \dots, N\}$ is

$$\mathbf{x}^i(t) = \mathrm{e}^{\mathbf{A}(t-t_k)}\mathbf{x}^i(t_k) + \bar{\mathbf{G}}_k(t)\mathbf{G}_k^{-1}(\mathbf{x}^0(t_k) - \mathrm{e}^{\mathbf{A}T_k}\, \mathbf{x}^i(t_k)); \tag{10}$$

(c) the control input $\mathbf{u}^i(t)$ of every agent $i \in \{1, \dots, N\}$ is equal to (4).

In the following, we give several remarks regarding the structural properties of the leader-follower algorithm of Theorem IV.1. First, it is worth to note here the interesting synchronization property that the leader-follower algorithm described in Theorem IV.1 has.

Corollary IV.1 (Followers' synchronization)**.** Consider the Leader-follower interaction described in Theorem IV.1. Then, $\mathbf{x}^i(t) = \mathbf{x}^j(t)$ for $t \in [t_1, \infty)$ and $\mathbf{u}^i(t) = \mathbf{u}^j(t)$ for $t \in (t_1, \infty)$, for every $i, j \in \{1, \dots, N\}$. Moreover, if $\mathbf{x}^i(0) = \mathbf{x}_0 \in \mathbb{R}^n$ for all $i \in \{1, \dots, N\}$, then these equalities also hold for $t \in [0, t_1]$ □

Next we observe the following minimum energy control property which follows from the statement (c) of Theorem IV.1 and the classical result in Lemma IV.1.

Corollary IV.2 (Minimum energy control in $[t_k, t_{k+1}]$)**.** Consider the Leader-follower interaction described in Theorem IV.1. Then, at each time interval $[t_k, t_{k+1}]$, $k \in \mathbb{Z}_{\geq 0}$, the control input $\mathbf{u}^i$ of each follower agent $i \in \{1, \dots, N\}$ is the minimum energy controller that transfers the agent from its current state $\mathbf{x}(t_k)$ to their desired state $\mathbf{x}(t_{k+1}) = \mathbf{x}^0(t_k)$. □

Remark IV.1 (Robustness to state perturbations)**.** We also observe that the leader-follower algorithm of Theorem IV.1 has robustness to perturbations similar to the well-known Model Predictive Control (MPC). Even though the controller implemented in each epoch $(t_k, t_{k+1}]$ is an open-loop control, since every agent inputs its state at time t_k as initial

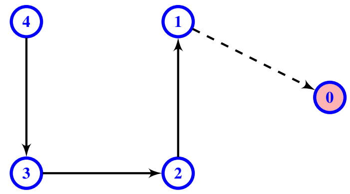

Fig. 2: Interaction topology of the leader-follower problem of Section V-A. Agent 0 is the leader.

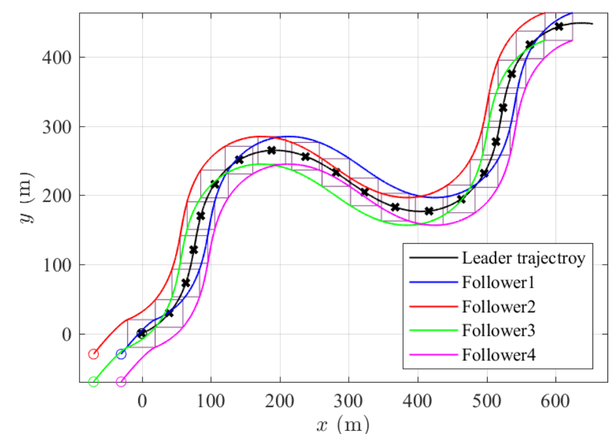

Fig. 3: The trajectories of the leader and the followers in the first numerical example.

condition to the controller, the algorithm can account for the slight perturbations in the agents final state $\mathbf{x}^i(t_{k+1})$ at the end of each epoch. □

Remark IV.2 (Tracking a priori known desired states with zero delay and design of arrival times)**.** Finally, we note that if the leader is virtual and the sampled states are some desired states that are known a priori to $\mathcal{N}_{\rm i}^0$ with desired arrival time in $\mathbb{R}_{>0}$, there will be no delay in tracking the leader's states. Furthermore, in cases that the arrival times is not specified one of the agents in $\mathcal{N}_{\rm i}^0$ (we refer to it as supper node) can design these arrival times to meet other optimality conditions or to avoid violating constraints such as input saturation. In case of input saturation, the fact that by virtue of statement (c) of Theorem IV.1 the form of input vector of the follower agents are known to be (4) can be instrumental to the supper node in design of arrival times. Our second demonstrative example in the proceeding section offers the details. □

V. Demonstrative Examples

In this section, we demonstrate use of our proposed algorithm in Theorem IV.1 in solving two leader-follower problems for mobile agents.

A. A leader tracking problem for a group of unicycle robots

In this demonstrative example, we use our leader-follower algorithm of Theorem IV.1 to solve a leader-follower problem for a group of unicycle robots

$$\begin{aligned} \dot{x}^i &= v^i \cos\theta^i, \\ \dot{y}^i &= v^i \sin\theta^i, \qquad i \in \{0,1,2,3,4\}, \\ \dot{\theta}^i &= \omega^i, \end{aligned} \tag{11}$$

where $v^i \in \mathbb{R}$ and $\omega^i \in \mathbb{R}$ are linear velocity and angular velocity of each agent i, respectively. Here, agents $\mathcal{V} = \{1,2,3,4\}$ are the follower agents, and agent 0 is the leader robot which moves with a known constant linear velocity of $v^0 = 100\,(\text{m/min})$ but unknown angular velocity ω^0. The interaction topology of the agents is shown in Fig. 2. Agent 1 obtains the states of the leader with a sampling rate of 2 per minute, i.e., $T_k = 0.5$ minutes, $k \in \mathbb{Z}_{\geq 0}$. The followers start at $\mathbf{x}^1(0) = [-30 \quad -30 \quad 0]^\top$, $\mathbf{x}^2(0) = [-70 \quad -30 \quad 0]^\top$, $\mathbf{x}^3(0) = [-70 \quad -70 \quad 0]^\top$, $\mathbf{x}^4(0) = [-30 \quad -70 \quad 0]^\top$ in a rectangular formation. The follower team wants to follow the leader in a rectangle formation that preserves the initial vertical and horizontal relative distances of the agents. To satisfy this objective, we first feedback

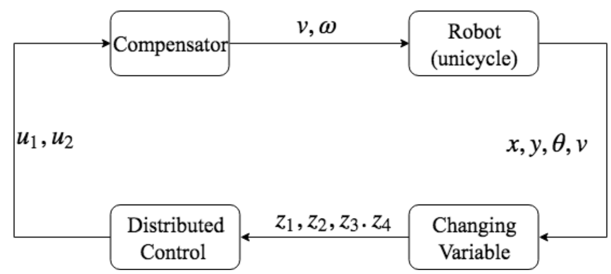

Fig. 4: The linearization procedure of unicycle.

linearize the dynamics of the followers and then implement our proposed leader-follower algorithm of Theorem IV.1 as described below. The results of implementing our leader-follower algorithm is shown in Fig. 3. The '$\times$' represents the sampled leader positions and the gray window shows the resulting formation of the followers at the sampling times.

It is well known that the unicycle dynamics is feedback linearizable. The linearization procedure is described in [22] and is shown in Fig. 4. For each follower agent $i \in \{1,2,3,4\}$, the feedback linearized dynamics consists of two decoupled second-order integral systems of

$$\begin{bmatrix} \dot{z}_1^i \\ \dot{z}_2^i \\ \dot{z}_3^i \\ \dot{z}_4^i \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} z_1^i \\ z_2^i \\ z_3^i \\ z_4^i \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{B}} \begin{bmatrix} u_1^i \\ u_2^i \end{bmatrix}, \tag{12}$$

with the changing variable

$$\begin{aligned} z_1^i &= x^i, \\ z_2^i &= v^i \cos\theta^i, \\ z_3^i &= y^i, \\ z_4^i &= v^i \sin\theta^i. \end{aligned} \tag{13}$$

The resulting dynamic compensator of each follower agent

$i \in \{1,2,3,4\}$ is

$$\dot{\eta}^i = u_1^i \cos\theta^i + u_2^i \sin\theta^i, \quad \eta^i(0) \in \mathbb{R},$$
$$v^i = \eta^i, \qquad (14)$$
$$\omega^i = \frac{u_2^i \cos\theta^i - u_1^i \sin\theta^i}{\eta^i}.$$

To follow the states of the leader, we assume that agent 1 constructs $z^0(t_k) = \begin{bmatrix} x^0(t_k) & v^0\cos(\theta(t_k)) & y^0(t_k) & v^0\sin(\theta(t_k)) \end{bmatrix}^\top$ from the leader's sampled state vector $x^0(t_k) = \begin{bmatrix} x^0(t_k) & y^0(t_k) & \theta^0(t_k) \end{bmatrix}^\top$. The follower agents share their feedback linearized states z^i. The follower agents use (9) to obtain $\mathbf{u}^i \in \mathbb{R}^2$, $i \in \{1,2,3,4\}$, of (12). Then, they obtain their inputs (v^i, ω^i), $i \in \{1,2,3,4\}$, from (14). Here, we note that the leaders dynamics is nonlinear and is not required to be feedback linearized. Given the initial location of the agents in Fig. 2, to preserve the initial rectangular formation at every $t = t_k$, the desired state of agent i is an offset with respect to the state of agent $j \in \mathcal{N}^i_{\text{out}}$. We define the offset parameter $\mathbf{p}^{ij} \in \mathbb{R}^4$ as the desired state offset of agent i at $t = t_k$ with respect to agent $j \in \mathcal{N}^i_{\text{out}}$. We set that the followers want to keep the sampled leader's states $\mathbf{x}^0(t_k)$ in the center of their rectangular formations at every $t = t_{k+1}$. Thereby, the offset parameters are $\mathbf{p}^{10} = [20 \quad 0 \quad 20 \quad 0]^\top$, $\mathbf{p}^{21} = [-40 \quad 0 \quad 0 \quad 0]^\top$, $\mathbf{p}^{32} = [0 \quad 0 \quad -40 \quad 0]^\top$ and $\mathbf{p}^{43} = [40 \quad 0 \quad 0 \quad 0]^\top$. Then, $\mathbf{x}^i(t_k) = \mathbf{x}^j(t_k) + \mathbf{p}^{ij}$, $j \in \mathcal{N}^i_{\text{out}}$, is achieved by adding the term $\mathbf{B}^\top \mathrm{e}^{\mathbf{A}^\top(t_{k+1}-t)}\, \mathbf{G}_k^{-1}\mathbf{p}^{ij}$ to the control (9). One can easily verify the validity of this approach in a similar way to the proof of Theorem IV.1. The details are omitted for brevity.

B. Reference state tracking for a group of second integrator dynamics with bounded inputs

We consider a group of 6 agents with second order integrator dynamics

$$\dot{\mathbf{x}}^i = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\mathbf{A}} \mathbf{x}^i + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\mathbf{B}} u^i, \quad -5 \le u^i \le 5, \qquad (15)$$

for $i \in \{1,\dots,6\}$. The interaction topology of these agents is shown in Fig. 5, where, agent 0 is the virtual leader that is defined more precisely below. Starting at initial conditions $\mathbf{x}^1(0) = [0 \quad 0]^\top$, $\mathbf{x}^2(0) = [2 \quad 0]^\top$, $\mathbf{x}^3(0) = [-2 \quad 0]^\top$, $\mathbf{x}^4(0) = [5 \quad 0]^\top$, $\mathbf{x}^5(0) = [10 \quad 0]^\top$, $\mathbf{x}^6(0) = [-10 \quad 0]^\top$, the leader-follower mission for this team is to traverse through the sequence of desired states $\mathbf{x}^{\mathrm{d}} = \{\mathbf{x}_1^{\mathrm{d}}, \mathbf{x}_2^{\mathrm{d}}, \mathbf{x}_3^{\mathrm{d}}, \mathbf{x}_4^{\mathrm{d}}\} = \left\{ \begin{bmatrix} 50 \\ 10 \end{bmatrix}, \begin{bmatrix} -50 \\ 10 \end{bmatrix}, \begin{bmatrix} 20 \\ 10 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$, which for privacy reason are only known to agent 1. The objective is to meet the sequence of desired states without violating any of the agents' control bounds.

In this problem setting, agent 1 is a supper node that also knows the initial starting state of all the followers in the team and has computational power to compute the arrival times to meet the team's objective as follows:

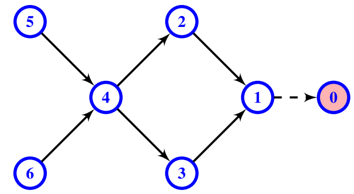

Fig. 5: Interaction topology with 6 agents. Agent **0** is the virtual leader.

- We note that by virtue of statement (c) of Theorem IV.1 the form of input vector of the follower agents are known to be (4). Since agent 1 knows $\mathbf{x}^i(t_0)$ for $i \in \mathcal{V}$, agent 1 can evaluate $u^i(t)$ of all the agents. Starting with $t_0^{\mathrm{d}} = 0$, agent 1 computes the arrival time at desired state $\mathbf{x}_1^{\mathrm{d}}$ from the process below

$$t_1^{\mathrm{d},i} = \operatorname{argmin} \int_{t_0^{\mathrm{d}}}^{t_1^{\mathrm{d},i}} \mathrm{d}\tau, \quad \text{s.t.} \qquad (16a)$$
$$-5 \le u^i(t) \le 5, \qquad (16b)$$

  where $u^i(t) = \mathbf{B}^\top \mathrm{e}^{\mathbf{A}^\top(t_1^{\mathrm{d},i}-t)}\, \mathbf{G}_0^{-1}(\mathbf{x}_0^{\mathrm{d}} - \mathrm{e}^{\mathbf{A}T_0}\,\mathbf{x}^i(0))$ with $T_0 = t_1^{\mathrm{d},i} - t_0^{\mathrm{d}}$. Then, the arrival time so that the agents input do not saturate over $(t_0^d, t_1^{\mathrm{d},i}]$ is set to $t_1^{\mathrm{d}} = \max\{t_1^{\mathrm{d},i}\}$.
- Due to Corollary IV.1, after first epoch, the agents inputs are equal to each other. Then, the remaining arrival time t_l^{d}, $l \in \{2,3,4\}$. Agent 1 computes these desired times from the optimization problem

$$t_{k+1}^{\mathrm{d}} = \operatorname{argmin} \int_{t_k^{\mathrm{d}}}^{t_{k+1}^{\mathrm{d}}} \mathrm{d}\tau, \quad \text{s.t.} \qquad (17a)$$
$$-5 \le u(t) \le 5, \qquad (17b)$$

  where $u(t) = \mathbf{B}^\top \mathrm{e}^{\mathbf{A}^\top(t_{k+1}^{\mathrm{d}}-t)}\, \mathbf{G}_k^{-1}(\mathbf{x}_{k+1}^{\mathrm{d}} - \mathrm{e}^{\mathbf{A}T_k}\,\mathbf{x}_k^{\mathrm{d}})$ with $T_k = t_{k+1}^{\mathrm{d}} - t_k^{\mathrm{d}}$, for $k \in \{1,2,3\}$.

The solution for this set of sequential optimal control problem is $t_1^{\mathrm{d}} = 6.7178$, $t_2^{\mathrm{d}} = 25.2061$, $t_3^{\mathrm{d}} = 30.1592$ and $t_4^{\mathrm{d}} = 40.4885$ seconds. Finally, agent 1 broadcasts the desired arrival times to the network. Broadcasting the reference states is not allowed due to privacy reasons. We note that these processes can be done offline. To match the notation in (9), at the implantation stage, we set $\mathbf{x}^0(t_{k-1}) = \mathbf{x}_k^{\mathrm{d}}$, $T_{k-1} = t_k^{\mathrm{d}} - t_{k-1}^{\mathrm{d}}$, and $t_k = t_{k-1} + T_{k-1}$, $k \in \{1,\dots,4\}$, where $t_0^{\mathrm{d}} = 0$. Figures 6 and 7 show all of the agents are going to achieve the reference states at the specified arrival times without delay. The '$\times$' in the figures marks the reference states. Figure 8 shows the control history of the agents. As seen the control inputs respect the saturation bounds 5 or -5. We can also observe that the followers' states and inputs, as predicted in Corollary IV.1, are all synchronized after the first epoch.

VI. Conclusion

In this paper, we have proposed a distributed leader-follower algorithm for multi-agent systems with an active leader with

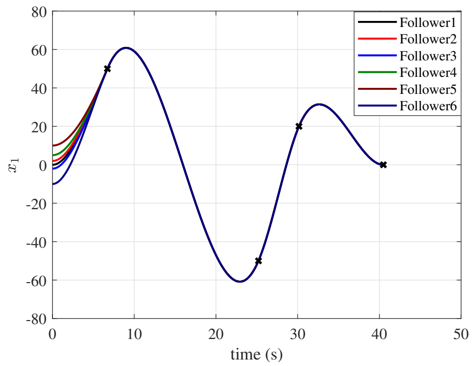

Fig. 6: The trajectories of x_1^i, $i \in \{1, \dots, N\}$, in the second numerical example.

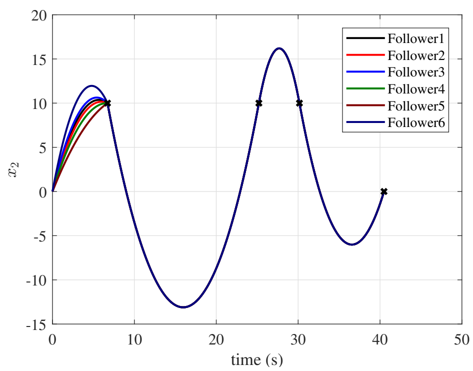

Fig. 7: The trajectories of x_2^i, $i \in \{1, \dots, N\}$, in the second numerical example.

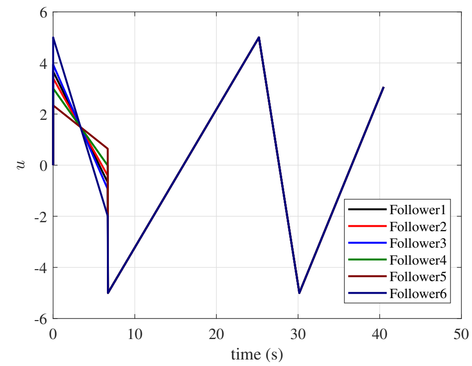

Fig. 8: Time history of control input u^i, $i \in \{1, \dots, N\}$, in the second numerical example.

unknown input. We have proved that our distributed leader-follower algorithm for the linear followers steers the group to be at the sampled states of the leader at the specified arrival times. We showed that the control input of each follower agent between the sampling times is a minimum energy control. We also showed that after the first sampling epoch, the states of all the follower agents are synchronized with each other. We demonstrated the application of our proposed algorithm for two leader-follower problems of mobile agents. Our future work includes extending our proposed algorithm to a heterogeneous group of follower agents and also output tracking.

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