A study on rate of convergence increase due to time delay for a class of linear systems

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Abstract— **This paper characterizes fully how time delay affects the rate of convergence of a class of linear time-delayed systems. Contrary to the prevailing intuition that links time delay with system sluggishness, we show that for specific ranges of time delay, faster response can be achieved in the presence of delay. Specifically, we determine exactly for what values of delay the rate of convergence of our system of interest increases with delay. We also prove that the ultimate bound on the maximum achievable rate of convergence via time delay is e (Euler's number) times the delay free rate. We demonstrate our results by studying the convergence rate of the Laplacian static average consensus algorithm in the presence of time delay.**

Keywords: Linear Time-delayed Systems, Relative Stability, Rate of Convergence, Lambert W Function, Accelerated Static Average Consensus

I. INTRODUCTION

Time delay in dynamical systems is most often presumed to result in sluggishness, performance loss and instability. However, some literature has exploited time delay to improve performance in some systems. For example, it is shown that delay is effective on stabilizing biology inspired systems [1]. Some other work introduced delay to a dynamical system in order to stabilize oscillatory behavior [2] tuning vibration absorbers [3], improving robustness [4], and steering system trajectory [5], [6]. In this paper, we study the positive effect of the time delay on increasing the stability margin and the rate of convergence of a class of linear time-delayed systems. Our study is based on characterizing the variation of the rate of convergence versus time delay using the Lambert W function. Lambert W function has been used to analyze time-delayed systems including stability analysis, eigenvalue assignment and obtaining the rate of convergence for different dynamical systems [7], [8], [9], [10].

The rate of convergence of a zero input response of a linear time-invariant system in the absence and presence of time delay is determined by the magnitude of the real part of the rightmost root of the system characteristic equation. For a system without delay the rightmost root is the rightmost eigenvalue of the system matrix. For linear systems with delay, various methods such as Lyapunov-based methods [11], [12], matrix measure [13], Riccati equation [14], Hopf bifurcation [15], α-stability [11] and pseudospectral and operator approximation techniques [16] are used to estimate the rate of convergence of time-delayed systems. For linear time-delayed systems with fixed time delay, [8] uses the Lambert W function to determine the exact location of the roots of the characteristic equation, and so the rate of convergence of the system. These methods only characterize the rate of convergence for a given time delay value. The well-known *continuity stability property* theorem [17, Proposition 3.1] for linear time-delayed systems with fixed delay states that if the delay free system is internally exponentially stable, then there always exists a range of delay for which the delayed system is exponentially stable. Moreover, for any value of the time delay beyond the one corresponding to the first time the rightmost root(s) of the characteristic equation are on the imaginary line the system is unstable. Even though the continuity stability property determines the *admissible* range of delay for which the system is exponentially stable, it does not discuss how the rate of convergence changes with time delay in this admissible range. Some work in the literature point to increase of stability margin and thus the rate of convergence of linear systems due to time delay [18], [19], [20], [21]. However, these results provide only sufficient conditions for ranges of time delay that result in increase of the rate of convergence for very specific linear systems. They also do not specify the maximum attainable rate of convergence due to the time delay.

In this paper, we investigate and fully characterize the effect of time delay on the rate of convergence of a class of linear dynamical systems. We determine the exact range of time delay for which the rate of convergence is higher than the rate of convergence of the delay free system. Moreover, we obtain the optimum time delay corresponding to the maximum attainable rate of convergence. We also prove that the ultimate bound on the maximum achievable rate of convergence via time delay is e times the delay free rate. An interesting application problem that can benefit from our results is feasibility study of use of delayed feedback (outdated information) to accelerate convergence of linear distributed algorithms for networked systems. To illustrate, we study the effect of time delay on the rate of convergence of the Laplacian static average consensus algorithm for a multi-agent system communicating over an undirected connected graph [22], [23]. The admissible range of time delay for which this algorithm converges is obtained in [22]. Here we use a numerical example to illustrate our results for this application. For reasons of space, some of the proofs are omitted and will appear elsewhere.

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Notation: We let $\mathbb{R}$, $\mathbb{R}_{>0}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Z}$, and $\mathbb{C}$ denote the set of real, positive real, non-negative real, integer, and complex numbers, respectively. The transpose of a matrix $\mathbf{A} \in \mathbb{R}^{n\times m}$ is $\mathbf{A}^\top$. The set of eigenvalues of matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is $\sigma(\mathbf{A})$. For $z \in \mathbb{C}$, $\mathrm{Re}(z)$, $\mathrm{Im}(z)$ are its real and imaginary parts, respectively.

II. Lambert W function

To develop our results we rely on some of the properties of the Lambert W function, which are reviewed below (c.f. [7], [24], [25]). For a given $z \in \mathbb{C}$, Lambert W function gives the solution of $s\,\mathrm{e}^s = z$, i.e., $s = W(z)$. Except for $z = 0$ which gives $W(0) = 0$, W is a multivalued function with the infinite number of solutions denoted by $W_k(z)$ with $k \in \mathbb{Z}$. Matlab or Mathematica have functions to evaluate $W_k(z)$. For any $z \in \mathbb{R}$, the value of all the branches of the Lambert W function except for some parts of branch 0 and branch -1 are complex (non-zero imaginary part). Zero branch of Lambert function, W_0 is of special interest in this paper. This branch is an injective function that has the following properties (see Fig. 1),

$$W_0(-\frac{1}{\mathrm{e}}) = -1,\ W_0(0) = 0, \tag{1a}$$

$$\mathrm{Re}(W_0(z)) > -1, \qquad z \in \mathbb{R}\backslash\{-\frac{1}{\mathrm{e}}\}, \tag{1b}$$

$$W_0(z) \in \mathbb{R}, \qquad z \in [-\frac{1}{\mathrm{e}}, \infty), \tag{1c}$$

$$\mathrm{Im}(W_0(z)) \in (-\pi, \pi)\backslash\{0\}, \qquad z \in \mathbb{C}\backslash[-\frac{1}{\mathrm{e}}, \infty). \tag{1d}$$

Lemma 2.1 (c.f. [7]): *For any $z \in \mathbb{C}$ we have*

$$\mathrm{Re}(W_0(z)) \geq \max\big\{\mathrm{Re}(W_k(z))|\ k \in \mathbb{Z}\backslash\{0\}\big\}. \tag{2}$$

The equality holds between branch 0 and -1 over $z \in \mathbb{R}_{\leq 0}$ where we have $\mathrm{Re}(W_0(z)) = \mathrm{Re}(W_{-1}(z))$.

Other properties of the Lambert W function which we use are

$$\frac{\mathrm{d}\,W(z)}{\mathrm{d}\,z} = \frac{1}{z + \mathrm{e}^{W(z)}}, \quad z \in \mathbb{C}\backslash\{\frac{1}{\mathrm{e}}\}, \tag{3a}$$

$$\lim_{z\to 0} \frac{W(z)}{z} = 1, \tag{3b}$$

and $W_0(x)$ around $x = 0$ is given by (convergence radius of $\frac{1}{\mathrm{e}}$)

$$W_0(x) = \sum\nolimits_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n. \tag{4}$$

III. Problem formulation

We study the effect of the fixed *time delay* $\tau \in \mathbb{R}_{>0}$ on the rate of convergence of the time-delayed system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t-\tau), \quad \sigma(\mathbf{A}) = \{\alpha_1, \cdots, \alpha_n\} \subset \mathbb{R}_{<0}, \tag{5a}$$

$$\mathbf{x}(t) = \boldsymbol{\phi}(t),\ t \in [-\tau, 0], \tag{5b}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state variable, $\mathbf{A} \in \mathbb{R}^{n\times n}$ is the system matrix, which is Hurwitz and has real eigenvalues,

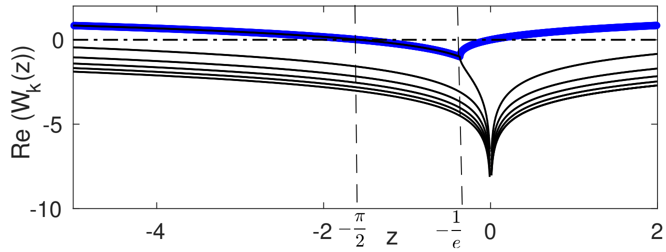


Fig. 1: $\mathrm{Re}(W_k(z))$ vs. $z \in [-5, 2]$ for $k = \{-5, -4, \cdots, 4, 5\}$: the solid thick blue line shows 0 branch while the black solid lines show the other branches. Some of the branches overlap.

and $\boldsymbol{\phi}(t)$ is a specified pre-shape function. We order the set of eigenvalues of $\mathbf{A}$ according to $|\alpha_1| \leq |\alpha_2| \leq \ldots \leq |\alpha_n|$. The trivial solution $\mathbf{x} \equiv \mathbf{0}$ of (5) is globally exponentially stable iff there exists a $\kappa \in \mathbb{R}_{>0}$ and an $\bar{\rho}_\tau \in \mathbb{R}_{>0}$ such trajectories of (5) satisfy

$$\|\mathbf{x}(t)\| \leq \kappa\,\mathrm{e}^{-\bar{\rho}_\tau\, t} \sup_{t\in[-\tau,0]} \|\mathbf{x}(t)\|, \quad t \in \mathbb{R}_{\geq 0}. \tag{6}$$

The exponential stability of (5) can be assessed in terms of the roots of its characteristic equation $\mathcal{F} : \mathbb{C} \to \mathbb{C}$ given by

$$\mathcal{F}(s) = \det\big(s\,\mathbf{I} - \mathbf{A}e^{-\tau\, s}\big) = \Pi_{i=1}^n (s - \alpha_i e^{-\tau\, s}). \tag{7}$$

Theorem 3.1 (c.f. [17]): *The linear time-delayed system* (5) *is exponentially stable if and only if*

$$\{s \in \mathbb{C}|\ \mathrm{Re}(s) \geq 0, \mathcal{F}(s) = 0\} = \{\}. \tag{8}$$

The characteristic equation (7) is transcendental and has an infinite number of roots in the complex plane. These roots can be obtained using Lambert W function. Let $p = \tau s$, then $(s - \alpha_i \mathrm{e}^{-\tau\, s}) = 0$ can be written as $p\,\mathrm{e}^p = \alpha_i\,\tau$, which specifies the roots of the characteristic equation (7) as

$$\big\{s \in \mathbb{C}\,|\, s = \frac{1}{\tau} W_k(\alpha_i\,\tau), \quad i \in \{1, \ldots, n\},\ k \in \mathbb{Z}\big\}. \tag{9}$$

As expected, as $\tau \to 0$ we recover the eigenvalues of $\mathbf{A}$ as the roots of the characteristic equation (7) (recall (3b)). Since system matrix $\mathbf{A}$ is Hurwitz, *the continuity stability property* theorem guarantees the existence of an $\epsilon \in \mathbb{R}_{>0}$ such that for all $\tau \in [0, \epsilon)$ the roots of the characteristic equation, i.e., (7), are all located strictly on the left hand side of the complex plane. The largest value of $\epsilon = \bar{\tau}$ beyond which the system becomes unstable is the minimum value of τ such that the characteristic equation (7) has roots on the imaginary axis. The following result gives the value of $\bar{\tau}$ for the system (5).

Lemma 3.1 (Admissible range for delay τ for the linear time-delayed system (5) [26]): *The time-delayed system* (5) *is exponentially stable if and only if* $\tau \in [0, \bar{\tau}) \subset \mathbb{R}_{\geq 0}$ *where*

$$\bar{\tau} = \min\{\bar{\tau}_i\}_{i=1}^n = \frac{\pi}{2\,|\alpha_n|}, \quad \bar{\tau}_i = \frac{\pi}{2\,|\alpha_i|}. \tag{10}$$

We refer to $[0, \bar{\tau})$ as the *admissible range* of delay. For any $\tau \in [\bar{\tau}, \infty)$, the system (5) becomes unstable.

Given (9) as the set of the roots of the characteristic equation (7), using Lemma 2.1, the rate of convergence of the system (5) is obtained as follows.

Lemma 3.2 (Rate of convergence of (5) for a delay in admissible range [27]): The exponential rate of convergence of the time-delayed system (5) for any $\tau \in [0, \bar{\tau})$, where $\bar{\tau}$ is given in (10), is given by

$$\rho_\tau = -\max\Big\{ \frac{1}{\tau} \mathrm{Re}(W_0(\alpha_i \tau)) \Big\}_{i=1}^{n}. \qquad (11)$$

Our objective in this paper is to show that for certain values of delay $\tau \in (0, \bar{\tau})$ it is possible for ρ_τ to be larger than the best rate of convergence of system (11) without delay. Recall that for a linear time-invariant system with no delay (i.e., when $\tau = 0$ in system (5)), the best rate of convergence is the absolute value of the real part of the rightmost eigenvalues of the system matrix (see [28]), which in case of system (5) is

$$\rho_0 = |\alpha_1|.$$

In what follows, we examine the variation of ρ_τ given in (11) with $\tau \in [0, \bar{\tau})$ to address the following questions: (a) for what $\mathbf{A}$ delay can lead to higher rate of convergence, (b) for what values of the delay we can have $\rho_\tau > \rho_0 = |\alpha_1|$ (c) what is the maximum value for ρ_τ and the corresponding maximizer $\tau^\star$. We use the following result in our developments below.

Lemma 3.3 (ρ_τ is a continuous function of τ): The rate of convergence ρ_τ of the linear time-delayed system (5) given by (11) is a continuous function of $\tau \in \mathbb{R}_{\geq 0}$,

Proof: Recall that for any given $\alpha \in \mathbb{R}$, $\mathrm{Re}(W_0(\alpha\tau))$ is a continuous function of $\tau \in \mathbb{R}_{\geq 0}$. Moreover, for any $\alpha \in \mathbb{R}_{<0}$, by virtue of (3b) we have $\lim_{\tau \to 0^-} \frac{\mathrm{Re}(W_0(\alpha\tau))}{\tau} = 1$. Therefore, for every α_i, $i \in \{1, \dots, n\}$, $\frac{\mathrm{Re}(W_0(\alpha_i\tau))}{\tau}$ is continuous over $\tau \in \mathbb{R}_{\geq 0}$. Then, the proof is deduced from the fact that the maximum of continuous functions is a continuous function (c.f. [29, Problem 1.2.13]). ∎

IV. Main result: Rate of convergence as a function of time delay

Our objective in this section is to investigate how the rate of convergence (11) of the liner time-delayed system (5) changes with $\tau \in \mathbb{R}_{>0}$ and identify conditions under which the rate of convergence increases by the delay. To compare the rate of convergence (11) to $\rho_0 = |\alpha_1|$, we define the *delay rate gain function* as follows

$$g(x) = \begin{cases} \frac{\mathrm{Re}(W_0(x))}{x}, & x \in \mathbb{R}_{<0}, \\ g(x) = 1, & x = 0. \end{cases} \qquad (12)$$

For any $\alpha \in \mathbb{R}_{<0}$ using delay rate gain we can write

$$\frac{1}{\tau} \mathrm{Re}(W_0(\alpha\tau)) = g(\alpha\tau)\, \alpha, \quad \tau \in \mathbb{R}_{>0}. \qquad (13)$$

Therefore, the rate of convergence (11) of the system (5) can be expressed also as

$$\rho_\tau = \min\big\{ g(\alpha_i \tau)\, |\alpha_i| \big\}_{i=1}^{n}, \quad \tau \in \mathbb{R}_{>0}. \qquad (14)$$

We refer to function $g : \mathbb{R}_{\leq 0} \to \mathbb{R}$ as the *delay rate gain*, since at each delay time $\tau \in \mathbb{R}_{>0}$, $g(\alpha_i \tau)$ is a measure of relative size of $\frac{1}{\tau} \mathrm{Re}(W_0(\alpha_i \tau))$ and α_i. In what follows, we first fully characterize the variations of $g(\alpha\tau)$ with respect to $\tau \in \mathbb{R}_{>0}$ for any given $\alpha \in \mathbb{R}_{<0}$. We use the results then to study the variation of ρ_τ with respect to $\tau \in (0, \bar{\tau})$.

A. Delay rate gain $g(x)$

In this section we study the delay rate gain (12), starting with the following result.

Lemma 4.1 (characterizing the solutions of $g(\alpha\tau) = 0$ and $g(\alpha\tau) = 1$): For a given $\alpha \in \mathbb{R}_{<0}$, the delay rate gain (12) satisfies:

$$\lim_{\tau \to 0} g(\alpha\, \tau) = 1, \qquad (15a)$$

$$(g(\alpha\, \bar{\tau}) = 0,\ \bar{\tau} \in \mathbb{R}_{>0}) \ \Leftrightarrow\ \bar{\tau} = \pi/2|\alpha|, \qquad (15b)$$

$$(g(\alpha\, \tilde{\tau}) = 1,\ \tilde{\tau} \in \mathbb{R}_{>0}) \ \Leftrightarrow\ \tilde{\tau} = \theta \cot(\theta)/|\alpha|, \qquad (15c)$$

where θ is the solution of $\mathrm{e}^{-\theta \cot(\theta)} = \cos(\theta)$ in $\theta \in (-\pi, \pi)$, which approximately is equal to ± 1.011.

The proof of this lemma follows from the proprieties of the Lambert W function listed in Section II. For example (15a) follows directly from (3b).

Following a similar argument to that of the proof of Lemma (3.3), next, we state the following result.

Corollary 4.1 ($g(\alpha\tau)$ is a continuous function of τ): For a given $\alpha \in \mathbb{R}_{<0}$, $g(\alpha\tau)$ is a continuous function of $\tau \in \mathbb{R}_{\geq}0$. ◁

For a given $\alpha \in \mathbb{R}$, using (3a) and (13), derivative of delay rate gain function with respect to time delay along $x = \alpha\tau \neq -\frac{1}{\mathrm{e}}$, $\tau \in \mathbb{R}_{>0}$ can be written as

$$\frac{d\, g(\alpha\tau)}{d\tau} = \frac{1}{\mathrm{Re}(\alpha)} \Big[\frac{-1}{\tau^2} \mathrm{Re}(W_0(\alpha\tau)) + \frac{1}{\tau} \mathrm{Re}\big(\frac{\alpha}{\alpha\tau + \mathrm{e}^{W_0(\alpha\tau)}}\big) \Big],$$

which can also be represented as $\frac{d\, g(\alpha\tau)}{d\tau} = -\frac{1}{\mathrm{Re}(\alpha)\tau^2} \mathrm{Re}\big(\frac{\alpha\tau\, W_0(\alpha\tau)}{\alpha\tau + e^{W_0(\alpha\tau)}}\big) = -\frac{1}{\mathrm{Re}(\alpha)\tau^2} \mathrm{Re}\big(\frac{W_0^2(\alpha\tau)}{1 + W_0(\alpha\tau)}\big)$. Let $W_0(\alpha\tau) = \mathsf{w} + \mathrm{i}\,\mathsf{u}$, where $\mathsf{u} \in (-\pi, \pi)$. Then, for $x = \alpha\tau \neq -\frac{1}{\mathrm{e}}$, we can write

$$\begin{aligned} \frac{d\, g(\alpha\tau)}{d\tau} &= -\frac{1}{\mathrm{Re}(\alpha)\tau^2} \frac{\mathsf{w}^3 + \mathsf{w}^2 - \mathsf{u}^2 + \mathsf{w}\mathsf{u}^2}{(\mathsf{w}+1)^2 + \mathsf{u}^2} \\ &= -\frac{1}{\mathrm{Re}(\alpha)\tau^2} \frac{(\mathsf{w}^2 + \mathsf{u}^2)\,\mathsf{w} + (\mathsf{w}^2 - \mathsf{u}^2)}{(\mathsf{w}+1)^2 + \mathsf{u}^2}. \end{aligned} \qquad (16)$$

Our objective in the following text is to show that given an $\alpha \in \mathbb{R}_{<0}$, there exists $\tau^\star$ such that for $\tau \in (0, \tau^\star) \subseteq [0, \bar{\tau})$ the rate of change of $g(\alpha\tau)$ with respect to τ is positive and as a result $g(\alpha\tau) > 1$.

Figure 2 shows the variation of $g(x)$ versus $x \in \mathbb{R}_{\leq 0}$ over $x \in [-2, 0]$. This plot reveals the following facts. (a) At $x = \bar{x} = -\frac{\pi}{2}$ we have $g(\bar{x}) = 0$ (as expected according to Lemma 4.1). For any $x \in (\bar{x}, 0)$, $g(x) > 0$ and for any $x \in (-\infty, \bar{x})$, $g(x) < 0$. (b) At $x = x^\star = -\frac{1}{\mathrm{e}}$, the maximum delay rate gain $g(x^\star) = \mathrm{e}$ is attained. For any $x \in (x^\star, 0)$, $g(x)$ monotonically decreases from e to 1, while for any $x \in (\bar{x}, x^\star)$, $g(x)$ increases monotonically from 0 to e. (c) Let $\tilde{x}$ be the non-zero solution of $g(\tilde{x}) = 1$, $x \in \mathbb{R}_{<0}$ (an approximate value of $\tilde{x}$ is -0.6334, see Lemma 4.1 for

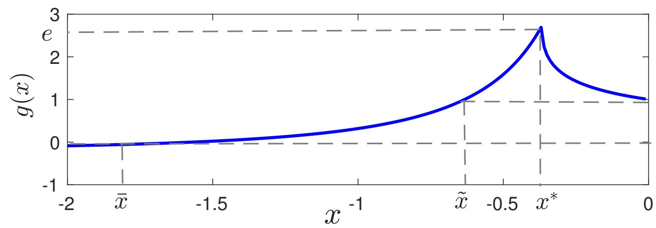


Fig. 2: delay rate gain versus $x \in \mathbb{R}_{<0}$. Points $x^{\star} = -\frac{1}{e}$, $\tilde{x} \approx -0.63$ and $\bar{x} = -\frac{\pi}{2}$, respectively, correspond to maximizer of $g(x)$, $g(x) = 1$ and $g(x) = 0$.

analytic characterization of $\tilde{x}$ using $\alpha = 1$ and $\tilde{x} = \alpha\tilde{\tau}$). Then, for any $x \in (\tilde{x}, 0)$ we have $g(x) > 1$. Also, for any $x \in [\bar{x}, \tilde{x}]$, $g(x)$ monotonically increases from 0 to 1. For a given $\alpha \in \mathbb{R}_{<0}$, let $x = \alpha\tau$. Then, using the aforementioned observations one can characterize the variation of $g(\alpha\tau)$ over $\tau \in (0, \bar{\tau})$ in comparison to 1 as in the following result. Nevertheless, for this result we also provide an alternative proof based on careful study of the derivative of $g(\alpha\tau)$ in (16).

Lemma 4.2 (variation of $g(\alpha\tau)$ for an $\alpha \in \mathbb{R}_{<0}$ and $\tau \in \mathbb{R}_{>0}$): *Let $\alpha \in \mathbb{R}_{<0}$ be given. Recall $\bar{\tau}$ and $\tilde{\tau}$ from Lemma 4.1. Let $\tau^{\star} = \frac{1}{e|\alpha|}$. Then, the followings hold:*

(a) *For any $\tau \in (0, \tau^{\star}) \subset (0, \bar{\tau})$, $g(\alpha\tau) > 1$, and $g(\alpha\tau)$ monotonically increases from 1 to e; $g(\alpha\tau^{\star}) = \mathrm{e}$; and for any $\tau \in (\tau^{\star}, \bar{\tau}) \subset (0, \bar{\tau})$, $g(\alpha\tau) > 0$, and $g(\alpha\tau)$ monotonically decreases from e to 0.*
(b) *For any $\tau \in (0, \tilde{\tau}) \subset (0, \bar{\tau})$, $g(\alpha\tau) > 1$; $g(\alpha\tilde{\tau}) = 1$; $g(\alpha\bar{\tau}) = 0$; for any $\tau \in (\tilde{\tau}, \bar{\tau})$, $0 < g(\alpha\tau) < 1$; and for $\tau \in (\bar{\tau}, \infty)$, $g(\alpha\tau) < 0$.*
(c) *The maximum value of $g(\alpha\tau)$ is e that is attained at $\tau = \tau^{\star}$.*

Proof: Because $W_0(-1/\mathrm{e}) = -1$, we obtain $g(\alpha\tau^{\star}) = \mathrm{Re}(W_0(-1/\mathrm{e}))/(-1/\mathrm{e}) = \mathrm{e}$. Next, note that from (15a), (15b) and Corollary 4.1, we know, respectively, that $\lim_{\tau\to 0} g(\alpha\,\tau) = 1$, $\tau = \bar{\tau}$ is the unique solution of $g(\alpha\tau) = 0$ for $\tau \in \mathbb{R}_{>0}$, and $g(\alpha\tau)$ is a continuous function of $\tau \in \mathbb{R}_{>0}$. Therefore, to complete proof of statement (a), we show next that $\frac{d\,g(x)}{d\tau} > 0$ for $\tau \in (0, \tau^{\star})$ and $\frac{d\,g(x)}{d\tau} < 0$ for $\tau \in (\tau^{\star}, \bar{\tau})$. For $x = \alpha\tau \in [-\frac{1}{\mathrm{e}}, 0]$, we have $W_0(x) \in \mathbb{R}$, and as such by setting $\mathsf{u} = 0$ from (16) we obtain

$$\frac{\mathrm{d}\,g(\alpha\tau)}{\mathrm{d}\tau} = \frac{1}{|\alpha|\,\tau^2}\,\frac{\mathsf{w}^2}{(\mathsf{w}+1)} > 0, \qquad \tau \in (0, \tau^{\star}). \tag{17}$$

Moreover, we have

$$\lim_{\tau\to\tau^{\star +}} \frac{\mathrm{d}\,g(\alpha\,\tau)}{\mathrm{d}\tau} = \lim_{\mathsf{w}\to -1^{+}} \mathrm{e}^2|\alpha|\,\frac{\mathsf{w}^2}{(\mathsf{w}+1)} = +\infty. \tag{18}$$

For any $x = \alpha\tau \in (-\infty, -\frac{1}{\mathrm{e}})$, $W_0(x) = \mathsf{w} + \mathsf{i}\,\mathsf{u}$ is a complex number with $\mathsf{u} \in (0, \pi)$ and satisfies

$$(\mathsf{w} + \mathsf{i}\,\mathsf{u})\mathrm{e}^{\mathsf{w}+\mathsf{i}\,\mathsf{u}} = \alpha\tau \Leftrightarrow \begin{cases} \mathrm{e}^{\mathsf{w}}\,(\mathsf{w}\cos(\mathsf{u}) - \mathsf{u}\sin(\mathsf{u})) = \alpha\tau, \\ \mathrm{e}^{\mathsf{w}}\,(\mathsf{u}\cos(\mathsf{u}) + \mathsf{w}\sin(\mathsf{u})) = 0. \end{cases}$$

Therefore, for $\alpha\tau \in (-\infty, -\frac{1}{\mathrm{e}})$, for which we always have $\mathsf{u} \neq 0$, we have $\mathsf{w} = -\mathsf{u}\cos(\mathsf{u})/\sin(\mathsf{u})$ and

$$\frac{\mathrm{d}\,g(\alpha\,\tau)}{\mathrm{d}\tau} = \frac{1}{|\alpha|\,\tau^2}\,\frac{\mathsf{u}^2(-\mathsf{u}\frac{\cos(\mathsf{u})}{\sin(\mathsf{u})} + \cos(2\,\mathsf{u}))}{(-\mathsf{u}\,\cos(\mathsf{u}){+}\sin(\mathsf{u}))^2 + \mathsf{u}^2\,\sin^2(\mathsf{u})}. \tag{19}$$

Because for $\alpha\tau \in [\frac{-\pi}{2}, -\frac{1}{\mathrm{e}})$ we have $\mathsf{u} \in (0, \frac{\pi}{2}]$, we can confirm that $-\mathsf{u}\frac{\cos(\mathsf{u})}{\sin(\mathsf{u})}{+}\cos(2\,\mathsf{u}) < 0$ and therefore, we obtain

$$\frac{\mathrm{d}\,g(\alpha\,\tau)}{\mathrm{d}\tau} < 0, \qquad \tau \in (\tau^{\star}, \bar{\tau}]. \tag{20}$$

From (19), we can also obtain that (invoking L'Hospital's rule [30, Theorem 5.5.2]) $\lim_{\tau\to\tau^{\star +}} \frac{\mathrm{d}\,g(\alpha\,\tau)}{\mathrm{d}\tau} = \lim_{\mathsf{u}\to 0} \frac{1}{|\alpha|\,\tau^{\star 2}}\,\frac{\frac{\mathsf{u}^2}{\sin^2(\mathsf{u})}(-\mathsf{u}\frac{\cos(\mathsf{u})}{\sin(\mathsf{u})}+\cos(2\mathsf{u}))}{(-\mathsf{u}\frac{\cos(\mathsf{u})}{\sin(\mathsf{u})}+1)^2+\mathsf{u}^2} = -\frac{5\,\mathrm{e}^2\,|\alpha|}{3}$. Statement (b) follows from (15a), (15c), (17), (20) and the continuity of $g(\alpha\tau)$ for $\tau \in \mathbb{R}_{>0}$. Statement (c) is deduced from statements (a) and (b), along with (18) and (19). ■

B. Rate of converge analysis in the presence of delay

Considering that the rate of convergence (11) of the time-delayed system (5) equivalently reads as (14), in this section we use the properties of the delay rate gain function to characterize fully how ρ_τ changes by delay. Our first result below states that any system that is represented by (5) experiences increase in its rate of convergence due to time delay. In what follows, $\tilde{\tau}_i$ is the non-zero solution of $g(\alpha_i\tilde{\tau}_i) = 1$, and $\bar{\tau}_i = \frac{\pi}{2|\alpha_i|}$ is the unique solution of $g(\alpha_i\bar{\tau}_i) = 0$, $i \in \{1, \dots, n\}$. Recall that for any $\alpha_i \in \sigma(\mathbf{A})$, $i \in \{1, \dots, n\}$, these delay values can be obtained by invoking Lemma 4.1.

Theorem 4.1 (Rate of convergence of the linear time-delayed system (5) can increase by time delay): *Consider the linear time-delayed system* (5) *and its rate of convergence* (11). *There always exists a $\hat{\tau} \in [\tilde{\tau}_n, \bar{\tau})$, such that $\rho_\tau > \rho_0$ for $\tau \in (0, \hat{\tau})$. Here $\bar{\tau}$ is given in* (10).

Proof: Since $\alpha_n \leq \alpha_{n-1} \leq \cdots \leq \alpha_1$, we have $\bar{\tau}_n \leq \bar{\tau}_{n-1} \leq \cdots \leq \bar{\tau}_1$ and $\tilde{\tau}_n \leq \tilde{\tau}_{n-1} \leq \cdots \leq \tilde{\tau}_1$ (see Lemma 4.1). Moreover, $\bar{\tau} = \bar{\tau}_n = \frac{\pi}{2|\alpha_n|}$ (c.f. Theorem 3.1). Since all the eigenvalues $\{\alpha_i\}_{i=1}^n$ of $\mathbf{A}$ in (5) are negative real numbers, according to the results of Lemma (4.2), for each α_i, $i \in \{1, \dots, n\}$, we have $g(\alpha_i\tau) > 1$, for $\tau \in (0, \tilde{\tau}_i) \subset (0, \bar{\tau}_i)$ and $g(\alpha_i\tau) < 1$ for $\tau \in (\tilde{\tau}_i, \bar{\tau}_i) \subset (0, \bar{\tau}_i)$. Consequently, we have $g(\alpha_i\tau)|\alpha_i| > |\alpha_i|$ for $\tau \in (0, \tilde{\tau}_i) \subset (0, \bar{\tau}_i)$, $i \in \{1, \dots, n\}$. Then, the proof of the statement follows from the definition of ρ_τ in (14) and its continuity with respect to $\tau \in \mathbb{R}_{>0}$ (see Lemma 3.3). ■

The following lemma shows that the rate of convergence ρ_τ depends on α_1 and α_n. This result paves the way to identify the exact value of $\hat{\tau}$ of Theorem 4.1 and also to determine the maximum attainable rate of convergence $\rho_\tau^{\star}$ and its corresponding maximizing time delay $\tau^{\star} \in (0, \bar{\tau})$. Here recall from Lemma 4.2 that for any $\alpha_i \in \sigma(\mathbf{A})$, the maximum value of $g(\alpha_i\tau)$ is e, which is obtained at $\tau_i^{\star} = \frac{1}{\mathrm{e}|\alpha_i|}$, $i \in \{1, \dots, n\}$. Note here that $\tau_n^{\star} \leq \tau_{n-1}^{\star} \leq \cdots \leq \tau_1^{\star}$.

Theorem 4.2 (ρ_τ depends only on α_1 and α_n): *The following assertions hold for the linear time-delayed system* (5)*:*

(a) $\rho_\tau = g(\alpha_1\tau)\,|\alpha_1| = -\frac{1}{\tau}\,\mathrm{Re}(W_0(\alpha_1\tau))$ *for any* $\tau \in (0,\tau_n^\star) \subset (0,\bar{\tau}]$,

(b) $\rho_\tau = \min\{g(\alpha_1\tau)\,|\alpha_1|, g(\alpha_n\tau)|\alpha_n|\}$ *for any* $\tau \in [\tau_n^\star,\tau_1^\star] \subset (0,\bar{\tau})$,

(c) $\rho_\tau = g(\alpha_n\tau)\,|\alpha_n| = -\frac{1}{\tau}\,\mathrm{Re}(W_0(\alpha_n\tau))$ *for any* $\tau \in (\tau_1^\star,\bar{\tau}) \subset (0,\bar{\tau})$,

where $\tau_1^\star = \frac{1}{\mathrm{e}|\alpha_1|}$ *and* $\tau_n^\star = \frac{1}{\mathrm{e}|\alpha_n|}$.

We are now ready to present our main result.

Theorem 4.3 (Rate of convergence of (5) with and without delay when $\{\alpha_i\}_{i=1}^n \subset \mathbb{R}_{<0}$): *Consider the linear time-delayed system* (5) *and its admissible delay bound* $\bar{\tau} = \frac{\pi}{2|\alpha_n|}$. *Then,*

(a) $\rho_\tau > \rho_0 = |\alpha_1|$ *if and only if* $\tau \in (0, \min\{\tilde{\tau}_1, \breve{\tau}\}) \subset (0,\bar{\tau})$ *where* $\breve{\tau}$ *is the unique non-zero solution of* $g(\alpha_n\breve{\tau}) = \frac{\alpha_1}{\alpha_n}$.

(b) *the maximum rate of convergence of*

$$\rho_\tau^\star = \mathrm{e}^{\frac{\arccos(\frac{1}{\gamma})}{\sqrt{\gamma^2-1}}}\,|\alpha_1|, \tag{21}$$

is attained at

$$\tau^\star = \frac{\arccos(\frac{1}{\gamma})}{\sqrt{\gamma^2-1}\,|\alpha_1|}\mathrm{e}^{-\frac{\arccos(\frac{1}{\gamma})}{\sqrt{\gamma^2-1}}} \in [\tau_n^\star,\tau_1^\star], \tag{22}$$

where $\tau_1^\star = \frac{1}{\mathrm{e}|\alpha_1|}$, $\tau_n^\star = \frac{1}{\mathrm{e}|\alpha_n|}$ *and* $\gamma = \frac{\alpha_n}{\alpha_1}$.

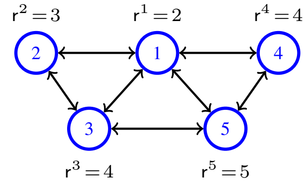
The next theorem provides an upper-bound on the maximum rate of convergence due to time delay for any system (5).

Theorem 4.4 (Upper-bound on the maximum achievable rate of convergence for the system (5)): *Consider the linear time-delayed system* (5) *and its admissible delay bound* $\bar{\tau} = \frac{\pi}{2|\alpha_n|}$. *The maximum achievable rate of convergence for this system satisfies* $\rho_\tau^\star \leq e\rho_0$ *(recall* (21)*).*

Proof: We note that the supremum value for $\arccos(\gamma)/\sqrt{1/\gamma^2-1}$ for $\gamma \in (0,1)$ is 1. Therefore, $\rho_\tau^\star$ in (21) is always less than or equal to $\mathrm{e}|\alpha_1| = \mathrm{e}\rho_0$, regardless of the value of $\alpha_i \in \mathbb{R}_{<0}$, $i \in \{1,\cdots,n\}$. ∎

V. Demonstrative Numerical Examples

A class of problems whose dynamical equation can be described by the linear-time delayed system (5) is the Laplacian average consensus algorithm over connected graphs in the presence of fixed communication delay (c.f. [23]). Due to space limitation, we describe the use of our results to study this application problem through a numerical example for a group of 5 agents whose interation topology is shown in Fig. 3. An edge between agent (node) i and agent (node) j means that these two agents can exchange information. The set of all agents that can send information to agent i are called its neighbors, denoted by $\mathcal{N}^i$. Let every agent in this network have a local reference value $\mathsf{r}^i \in \mathbb{R}$, $i \in \{1,\cdots,N\}$. The static average consensus problem consists of designing a distributed algorithm that enables each agent to obtain



$$\mathbf{L} = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & -1 \\ -1 & 0 & 0 & 2 & -1 \\ -1 & 0 & -1 & -1 & 3 \end{bmatrix}$$

Fig. 3: A connected undirected graph of 5 nodes.

$\frac{1}{N}\sum_{j=1}^N \mathsf{r}^j$ by using the information it receives only from its neighbors. Since the network in Fig. 3 is connected, i.e., there is a path from each agent to all the other agents in the network, the Laplacian dynamics

$$\dot{x}^i(t) = \sum\nolimits_{j\in\mathcal{N}^i}(x^j(t) - x^i(t)),\;\; x^i(0) = \mathsf{r}^i,\;\; i \in \{1,\cdots,N\},$$

is guaranteed to satisfy $x^i \to \frac{1}{N}\sum_{j=1}^N \mathsf{r}^j$, as $t \to \infty$ (c.f. [23]). Please see [31] for graph related terminologies and definitions. Using the aggregated state vector $\mathbf{x} = [x^1,\cdots,x^N]^\top$ the compact form of the dynamics above in the presence of delay is

$$\dot{\mathbf{x}} = -\mathbf{L}\,\mathbf{x}(t-\tau), \tag{23}$$
$$x^i(t) \in \mathbb{R}, \quad t \in [-\tau, 0),\;\; x^i(0) = \mathsf{r}^i,\;\; i \in \{1,\ldots,N\}.$$

where the network Laplacian matrix $\mathbf{L}$ is given in Fig. 3. The Laplacian matrix of a connected graph is a positive semi-definite matrix which has only a simple zero eigenvalue with the corresponding eigenvector $\mathbf{1}_N$, , where $\mathbf{1}_N$ is the vector of all ones of size N. For the graph shown in Fig. 3 the Laplacian matrix satisfies $\mathrm{rank}(\mathbf{L}) = 4$, $\mathbf{1}_5^\top\mathbf{L} = \mathbf{0}$ and $\mathbf{L}\mathbf{1}_5 = \mathbf{0}$. Next, consider the change of variable $\mathbf{y} = \mathbf{T}^\top\mathbf{x}$ with $\mathbf{T} = \begin{bmatrix}\frac{1}{\sqrt{5}}\mathbf{1}_5 & \mathbf{R}\end{bmatrix}$, where $\mathbf{R}$ is such that $\mathbf{T}^\top\mathbf{T} = \mathbf{T}\mathbf{T}^\top = \mathbf{I}_N$. Then, the Laplacian dynamics (23) can be represented in the following equivalent form

$$\dot{y}_1(t) = 0, \tag{24a}$$
$$\dot{\mathbf{y}}_{2:N}(t) = -(\mathbf{R}^\top\mathbf{L}\mathbf{R})\mathbf{y}_{2:N}(t-\tau). \tag{24b}$$

where $\mathbf{y} = [y_1^\top\;\mathbf{y}_{2:N}^\top]^\top$. The matrix $-\mathbf{R}^\top\mathbf{L}\mathbf{R}$ is Hurwitz with $\sigma(-\mathbf{R}^\top\mathbf{L}\mathbf{R}) = \{\alpha_i\}_{i=1}^n = \{-1.58,-3,-4.4,-5\}$. The rate of convergence of the Laplacian algorithm is defined by rate of convergence of (24b). Here, invoking Lemma 3.1 and Theorem 4.3 we predict that $\bar{\tau} = \frac{\pi}{2|-5|} \approx 0.31$, $\tilde{\tau} = \min\{\tilde{\tau}_1 = 0.4, \breve{\tau} = 0.2\} \approx 0.2$ and $\tau^\star = \arccos(\frac{1}{5/1.58})/\left(\sqrt{(5/1.58)^2-1}\mathrm{e}^{\arccos(\frac{1}{5/1.58})/\sqrt{(5/1.58)^2-1}}|-1.58|\right) \approx 0.17$ seconds, which match exactly the values one reads on the plot of ρ_τ $\tau \in \mathbb{R}_{>0}$ in Fig. 4. Figure 4 also highlights the dependency of ρ_τ on α_1 and $\alpha_{n=4}$ as in Theorem 4.2. Figure 5 shows the norm of response of (24b) in logarithmic scale versus time for different values of time delay. As seen in the figure, the rate of convergence $\rho_\tau^\star = 2.4$ corresponding to $\tau = \tau^\star = 0.17$ is greater than $\rho_0 = 1.58$. For $\tau = 0.3$ which is close to $\bar{\tau} = 0.31$, the rate of convergence is very slow.

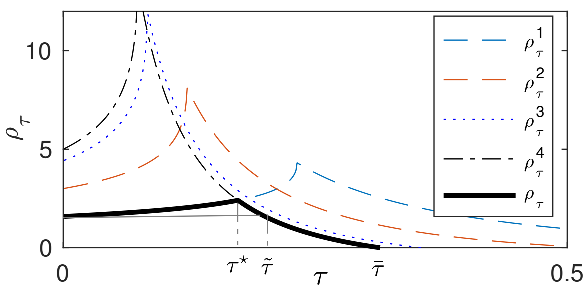


Fig. 4: Rate of convergence versus time delay for the network depicted in Figure 3.

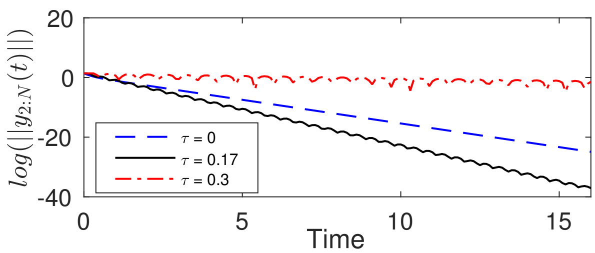


Fig. 5: Norm of time response of the system (24b) in logarithmic scale for different value of the time delay.

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VI. Conclusion

We examined the effect of time delay on the rate of convergence of a class of linear time-delayed systems. We showed that for this class, the rate of convergence always increases with time delay for some ranges of the admissible time delay range. We obtained this range along with the maximum attainable rate of convergence and its corresponding time delay value. We also showed that the ultimate bound on the maximum attainable rate due to time delay is e (Euler's number) times the delay free rate. We demonstrated our results via a numerical example that discusses the effect of delay on the rate of convergence of the Laplacian static average convergence algorithm executed over a network of 5 agents. Our future work is focused on expanding our results to a wider class of linear time-delayed systems. We will also investigate the benefits of use of delayed feedback (outdated information) to accelerate convergence of linear distributed algorithms for networked systems.

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